

Structure-preserving numerical method for a class of stochastic Poisson systems*

LIU Qianqian, WANG Lijin[†]

(School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing 100049, China)

(Received 24 November 2021; Revised 28 February 2022)

Liu Q Q, Wang L J. Structure-preserving numerical method for a class of stochastic Poisson systems[J]. Journal of University of Chinese Academy of Sciences, 2024, 41(3):298-305. DOI:10.7523/j.ucas.2022.015.

Abstract In this paper, we consider the structure-preserving numerical simulation of a class of stochastic Poisson systems, i. e. the stochastic Lotka-Volterra systems. We propose a stochastic Poisson integrator for the systems which can preserve the Poisson structure and the Casimir functions of the systems, and prove that the numerical integrator has root mean-square convergence order 1. Numerical experiments are performed to verify the theoretical results.

Keywords stochastic Poisson systems; Lotka-Volterra systems; Stratonovich SDEs; Poisson structure; Casimir functions

CLC number: O241.8 **Document code:** A **DOI:** 10.7523/j.ucas.2022.015

一类随机泊松系统的保结构数值方法

刘倩倩, 王丽瑾

(中国科学院大学数学科学学院, 北京 100049)

摘要 考虑一类随机泊松系统, 称为随机 Lotka-Volterra 系统的保结构数值模拟。为该类系统提出一种随机泊松积分分子, 并证明了该积分分子具有一阶均方收敛阶。数值实验对理论结果进行了验证。

关键词 随机泊松系统; Lotka-Volterra 系统; Stratonovich 型随机微分方程; Poisson 结构; Casimir 函数

Stochastic Poisson systems are defined as the form^[1]

$$\begin{cases} d\mathbf{y}(t) = \mathbf{B}(\mathbf{y}(t)) (\nabla H_0(\mathbf{y}(t)) dt + \\ \sum_{r=1}^s \nabla H_r(\mathbf{y}(t)) \circ dW_r(t)), \\ \mathbf{y}(t_0) = \mathbf{y}_0, \end{cases} \quad (1)$$

where $\mathbf{y}(t) = (y_1, \dots, y_n)^T$, $\mathbf{B}(\mathbf{y}) = (b_{ij}(\mathbf{y}))_{n \times n}$ is a skew-symmetric matrix, and satisfies the condition

$$\sum_{v=1}^n \left(\frac{\partial b_{ij}(\mathbf{y})}{\partial y_v} b_{vk}(\mathbf{y}) + \frac{\partial b_{jk}(\mathbf{y})}{\partial y_v} b_{vi}(\mathbf{y}) + \frac{\partial b_{ki}(\mathbf{y})}{\partial y_v} b_{vj}(\mathbf{y}) \right) = 0, \quad (2)$$

* Supported by the National Natural Science Foundation of China (11971458, 11471310, 11071251)

[†] Corresponding author, E-mail: ljwang@ucas.ac.cn

for all $i, j, k \in \{1, \dots, n\}$. $H_r(\mathbf{y}), r = 0, \dots, s$ are smooth scalar functions. $\mathbf{W}(t) = (W_1(t), \dots, W_s(t))$ is an s -dimensional standard Wiener process, and \circ stands for the Stratonovich product. We assume that $\mathbf{B}(\mathbf{y})$ is of constant rank $2m = n - l$, $0 \leq l < n$.

A skew-symmetric matrix $\mathbf{B}(\mathbf{y})$ with property (2) can define the Poisson bracket $\{F, G\}(\mathbf{y})$ of two smooth functions $F(\mathbf{y})$ and $G(\mathbf{y})$ as Refs. [1-2]

$$\{F, G\}(\mathbf{y}) := \sum_{i,j=1}^n \frac{\partial F(\mathbf{y})}{\partial y_i} b_{ij}(\mathbf{y}) \frac{\partial G(\mathbf{y})}{\partial y_j},$$

or equivalently in the form

$$\{F, G\}(\mathbf{y}) := \nabla F(\mathbf{y})^T \mathbf{B}(\mathbf{y}) \nabla G(\mathbf{y}).$$

It is known that almost surely the phase flow $\varphi_{t, \omega} : \mathbf{y}_0 \rightarrow \mathbf{y}(t)$ of the stochastic Poisson system (1) preserves the Poisson structure^[1],

$$\frac{\partial \mathbf{y}(t)}{\partial \mathbf{y}_0} \mathbf{B}(\mathbf{y}_0) \frac{\partial \mathbf{y}(t)}{\partial \mathbf{y}_0}^T = \mathbf{B}(\mathbf{y}(t)), \forall t \geq t_0. \tag{3}$$

A function $C(\mathbf{y})$ is called Casimir function of the system (1) if

$$\nabla C(\mathbf{y})^T \mathbf{B}(\mathbf{y}) = 0, \text{ for all } \mathbf{y}.$$

It is not difficult to prove that each Casimir function $C(\mathbf{y})$ is an invariant for the system (1)^[1].

$$\text{If } n = 2m \text{ and } \mathbf{B}(\mathbf{y}) = \begin{pmatrix} \mathbf{0} & -\mathbf{I}_m \\ \mathbf{I}_m & \mathbf{0} \end{pmatrix}, \text{ where } \mathbf{I}_m$$

denotes an m -dimensional identity matrix, the stochastic Poisson system (1) degenerates to the stochastic canonical Hamiltonian system. The phase flow of a stochastic Hamilton system preserves the symplectic structure^[3-4]. The Poisson structure of stochastic Poisson systems is an extension of the symplectic structure. For stochastic Poisson systems, there arised some numerical results in recent years. For a special class of stochastic Poisson systems, numerical methods are proposed in Ref. [5] and Ref. [6] respectively, where the method proposed in Ref. [5] can exactly preserve quadratic Casimir functions and the energy, while the method given in Ref. [6] is energy-preserving with the ability of arriving any prescribed order. In Ref. [7], for stochastic Poisson systems of even dimensions, the structure-preserving Runge-Kutta

and partitioned Runge-Kutta methods are proposed. The Darboux-Lie theorem and symplectic methods are used to obtain structure-preserving numerical schemes for stochastic Poisson systems in Ref. [1].

In this paper, we consider the following stochastic Lotka-Volterra (L-V) system with Stratonovich white noise^[5,8]

$$\begin{cases} d\mathbf{y}(t) = \mathbf{B}(\mathbf{y}(t)) \nabla H(\mathbf{y}(t)) (dt + c \circ dW(t)), \\ \mathbf{y}(t_0) = \mathbf{y}_0, \end{cases} \tag{4}$$

with

$$\begin{aligned} H(\mathbf{y}) &= \sum_{i=1}^n \beta_i y_i - p_i \ln y_i, \\ \mathbf{B}(\mathbf{y}) &= \text{diag}(y_1, \dots, y_n) \mathbf{B} \text{diag}(y_1, \dots, y_n), \end{aligned} \tag{5}$$

where $\beta_i \neq 0 (i = 1, \dots, n)$, \mathbf{B} is a skew-symmetric constant matrix and $c > 0$ measures the size of the perturbation. Moreover we assume the Assumption 2.1 holds. It was proved in Ref. [8] that, under Assumption 2.1, for any given $\mathbf{y}_0 > \mathbf{0}$, the system (4)-(5) has a unique solution which is positive for all $t \geq t_0$ almost surely. Hereby and in the sequel we write a vector $\mathbf{a} > \mathbf{0}$ to mean that each element of \mathbf{a} is positive. Meanwhile, the system (4)-(5) is a stochastic Poisson system, with Casimir functions $C(\mathbf{y}) = \alpha_1 \ln y_1 + \dots + \alpha_n \ln y_n$, $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)^T \in \text{Ker } \mathbf{B}$ and Hamiltonian $H(\mathbf{y})$ which is smooth for $\mathbf{y} > \mathbf{0}$.

In this paper, we transform the stochastic L-V system to a stochastic Hamiltonian system by coordinate transformation. Then we apply the midpoint method, which is a symplectic scheme, to the stochastic Hamiltonian system and by using the inverse transformation to obtain the numerical scheme for the stochastic L-V system. This method is shown to preserve the Poisson structure and Casimir functions of the stochastic L-V system. We further prove that the method is of root mean-square convergence order 1.

1 A Poisson integrator

1.1 The Darboux-Lie theorem

The Darboux-Lie theorem is as follows.

Theorem 1.1 (see Ref. [2]) Suppose that

the matrix $\mathbf{B}(\mathbf{y})$ defines a Poisson bracket and is of constant rank $n - l = 2m$ in a neighborhood of $\mathbf{y}_0 \in \mathbb{R}^n$. Then there exist functions $P_1(\mathbf{y}), \dots, P_m(\mathbf{y}), Q_1(\mathbf{y}), \dots, Q_m(\mathbf{y})$, and $C_1(\mathbf{y}), \dots, C_l(\mathbf{y})$ satisfying

$$\begin{cases} \{P_i, P_j\} = 0, & \{P_i, Q_j\} = -\delta_{ij}, & \{P_i, C_v\} = 0, \\ \{Q_i, P_j\} = \delta_{ij}, & \{Q_i, Q_j\} = 0, & \{Q_i, C_v\} = 0, \\ \{C_k, P_j\} = 0, & \{C_k, Q_j\} = 0, & \{C_k, C_v\} = 0, \end{cases} \quad (6)$$

for $i, j \in \{1, \dots, m\}$, $v \in \{1, \dots, l\}$ on a neighborhood of \mathbf{y}_0 . The gradients of P_i, Q_j, C_v , ($i, j \in \{1, \dots, m\}$, $v \in \{1, \dots, l\}$) are linearly independent, so that the $\mathbb{R}^n \rightarrow \mathbb{R}^n$ mapping $\boldsymbol{\theta}(\mathbf{y}) : \mathbf{y} \rightarrow (P_1(\mathbf{y}), \dots, P_m(\mathbf{y}), Q_1(\mathbf{y}), \dots, Q_m(\mathbf{y}), C_1(\mathbf{y}), \dots, C_l(\mathbf{y}))$ constitutes a local change of coordinates to canonical form.

If we denote $\bar{\mathbf{y}} = (\mathbf{Z}(\mathbf{y})^T, \mathbf{C}(\mathbf{y})^T)^T$, where

$$\begin{aligned} \mathbf{Z}(\mathbf{y}) &= (\mathbf{P}(\mathbf{y})^T, \mathbf{Q}(\mathbf{y})^T)^T, \\ \mathbf{P}(\mathbf{y}) &= (P_1(\mathbf{y}), \dots, P_m(\mathbf{y}))^T, \\ \mathbf{Q}(\mathbf{y}) &= (Q_1(\mathbf{y}), \dots, Q_m(\mathbf{y}))^T, \\ \mathbf{C}(\mathbf{y}) &= (C_1(\mathbf{y}), \dots, C_l(\mathbf{y}))^T, \end{aligned}$$

by the Darboux-Lie theorem, the system (4) can be transformed to^[1]

$$\begin{cases} d\mathbf{Z} = \mathbf{J}^{-1} \nabla_{\mathbf{z}} K(\mathbf{Z}, \mathbf{C})(dt + c \circ dW(t)), \\ d\mathbf{C} = 0, \end{cases} \quad (7)$$

which is a stochastic Hamiltonian system (SHS) with constant parameter vector \mathbf{C} , and $K(\mathbf{Z}, \mathbf{C}) =$

$$K(\bar{\mathbf{y}}) = H(y), \mathbf{J}^{-1} = \begin{pmatrix} \mathbf{0} & -\mathbf{I}_m \\ \mathbf{I}_m & \mathbf{0} \end{pmatrix}. K \text{ is smooth owing to the smoothness of } H.$$

According to the results in Ref. [1], applying a symplectic scheme to SHS (7) and then transforming the scheme back to system (4) by the inverse transformation $\mathbf{y} = \boldsymbol{\theta}^{-1}(\bar{\mathbf{y}})$, one gets a stochastic Poisson integrator for the stochastic Poisson system (4), which can preserve both the Poisson structure and the Casimir functions of system (4) almost surely.

1.2 The numerical scheme

As mentioned above, stochastic Poisson integrators can be obtained by using symplectic methods to the SHS arising from the coordinate transformation. It is well-known that the midpoint

method is a symplectic method for SHS^[9]. Therefore we propose a numerical scheme for system (4) derived from the transformation of the midpoint scheme, which we call the transformed midpoint (TM) method in the sequel.

The TM method:

- (a) By the Darboux-Lie theorem, we find a coordinate transformation $\boldsymbol{\theta}(\mathbf{y}) : \mathbf{y} \rightarrow \bar{\mathbf{y}} = (\mathbf{Z}(\mathbf{y})^T, \mathbf{C}^T)^T$, which transforms system (4) with initial value \mathbf{y}_0 to SHS (7) with initial value $\mathbf{Z}_0 = (\mathbf{P}(\mathbf{y}_0)^T, \mathbf{Q}(\mathbf{y}_0)^T)^T$.
- (b) Apply the midpoint method to SHS (5)

$$\mathbf{Z}_{j+1} = \mathbf{Z}_j + \mathbf{J}^{-1} \nabla_{\mathbf{z}} K \left(\frac{\mathbf{Z}_{j+1} + \mathbf{Z}_j}{2}, \mathbf{C} \right) (h + c \Delta \hat{W}_j), \quad (8)$$

where h is the time step and $\Delta \hat{W}_j$ ($j = 0, 1, \dots$) are truncated Wiener increments (explained below).

- (c) Solve equation (6) for \mathbf{Z}_{j+1} , then use the inverse transformation $\mathbf{y}_{j+1} = \boldsymbol{\theta}^{-1}(\bar{\mathbf{y}}_{j+1})$ with $\bar{\mathbf{y}}_{j+1} = (\mathbf{Z}_{j+1}^T, \mathbf{C}^T)^T$ to get the numerical solution \mathbf{y}_{j+1} ($j = 0, 1, \dots$) for system (4).

In equation (8), $\Delta \hat{W}_j = \sqrt{h} \xi_j$ is a truncation of $\Delta W_j = \sqrt{h} \xi_j$, where $\xi_j \sim \mathcal{N}(0, 1)$ and

$$\xi_j = \begin{cases} \xi_j, & \text{if } |\xi_j| \leq A_h, \\ A_h, & \text{if } \xi_j > A_h, \\ -A_h, & \text{if } \xi_j < -A_h, \end{cases}$$

where $A_h := \sqrt{2k \lceil \ln(h) \rceil}$ (for an integer $k \geq 1$)^[9]. Moreover, one has the following estimates^[9-10]:

$$\begin{aligned} \mathbb{E} [\xi_j - \zeta_j]^2 &\leq h^2, & (9) \\ 0 &\leq \mathbb{E} [\xi_j^2 - \zeta_j^2] = (1 + 2\sqrt{2k \lceil \ln(h) \rceil}) h^k, \\ \mathbb{E} [\xi_j^2 - \zeta_j^2]^2 &\leq 27h. \end{aligned}$$

As discussed in Ref. [9], such a truncation is a remedy to fix the issue in implementing an implicit scheme caused by the unboundedness of ΔW_j ($j = 0, 1, \dots$). By choosing sufficiently large parameter k , the truncation error can be merged into the error of the numerical scheme, and for a method of root mean-square order l , it is sufficient to choose $k \geq 2l$. Here in this paper we take $k = 4$ in A_h which is sufficient for our discussion.

Remark 1.1 With the truncation of ΔW_j , the

fixed point iteration by solving equation (8) for \mathbf{Z}_{j+1} can converge for sufficiently small h , similar to the discussion in Refs. [9, 11].

Remark 1.2 The Darboux-Lie transformation $\boldsymbol{\theta}(\mathbf{y})$ varies for different concrete problems, usually found by solving the equations in (6).

Then by the discussion in section 1.1 we have the following conclusion.

Theorem 1.2 The TM method preserves the Poisson structure and Casimir functions of the Stochastic L-V System (4)-(5).

2 Convergence order

In this section we will prove the root mean-square convergence order of the TM method.

Assumption 2.1 (see Ref. [8]) Assume that for the parameters $\boldsymbol{\beta} = (\beta_1, \dots, \beta_n)^T, \mathbf{p} = (p_1, \dots, p_n)^T$ of the system (4)-(5), there exists a real number $s_0 \in \mathbb{R}$ and a vector $\boldsymbol{\alpha} \in \text{Ker} \mathbf{B}$ such that

$$\begin{cases} s_0 \boldsymbol{\beta} > 0, \\ -s_0 \mathbf{p} + \boldsymbol{\alpha} < 0. \end{cases} \quad (10)$$

Lemma 2.1 (see Ref. [8]) Under Assumption 2.1, for any given initial value $\mathbf{y}_0 > \mathbf{0}$, the exact solution of system (4)-(5) is almost surely bounded.

Based on Lemma 2.1, we can prove the boundedness of the numerical solution of the TM method. We assume the numerical approximation is performed on the time interval $t \in [t_0, T]$, and $t_N = T$.

Theorem 2.1 Under Assumption 2.1, given $\mathbf{y}_0 > \mathbf{0}$, the numerical solution $\{\mathbf{y}_j\} (j = 1, \dots, N)$ arising from the TM method is bounded almost surely.

Proof Given $\mathbf{y}_0 > \mathbf{0}$, by Lemma 2.1, the exact solution $y(t)$ of system (4)-(5) is almost surely bounded. Since $\boldsymbol{\theta}(\mathbf{y})$ is continuous by the Darboux-Lie theorem ($\boldsymbol{\theta}'(\mathbf{y})$ exists and invertible), the solution of system (7) is almost surely bounded. This means there are constants L_1, L_2 such that $(\mathbf{P}(\mathbf{y})^T, \mathbf{Q}(\mathbf{y})^T)^T \in [L_1, L_2]^{2m}$ almost surely. For a vector $\mathbf{a} \in \mathbb{R}^{2m}$, we say $\mathbf{a} \in [L_1, L_2]^{2m}$ to mean that each element of \mathbf{a} is in $[L_1, L_2]$.

For $\mathbf{Z}_0 \in [L_1, L_2]^{2m}$, we define

$$\boldsymbol{\phi}(\mathbf{Z}) = \mathbf{Z}_0 + \mathbf{J}^{-1} \nabla_z K \left(\frac{\mathbf{Z} + \mathbf{Z}_0}{2}, \mathbf{C} \right) (h + c \Delta \hat{W}_0). \quad (11)$$

Then the scheme (6) can be written as

$$\mathbf{Z}_1 = \boldsymbol{\phi}(\mathbf{Z}_1), \quad (12)$$

when $j = 0$. We can use the fixed-point iteration method to solve (12). Select the initial value z of iteration that satisfies $0 < \|z - \mathbf{Z}_0\| < 1$, thus $z \in [L_1 - 1, L_2 + 1]^{2m} \setminus \mathbf{Z}_0$. We have

$$\begin{aligned} \|\boldsymbol{\phi}(z) - \mathbf{Z}_0\| &= \|\mathbf{J}^{-1} \nabla K \left(\frac{z + \mathbf{Z}_0}{2}, \mathbf{C} \right) (h + c \zeta_0 \sqrt{h})\| \\ &= \|\mathbf{J}^{-1} \nabla K \left(\frac{z + \mathbf{Z}_0}{2}, \mathbf{C} \right)\| (h + c |\zeta_0| \sqrt{h}) \end{aligned}$$

Since $\frac{z + \mathbf{Z}_0}{2} \in [L_1 - 1, L_2 + 1]^{2m}$, and the function $\mathbf{J}^{-1} \nabla K(z, \mathbf{C})$ is continuous, we have

$$\|\boldsymbol{\phi}(z) - \mathbf{Z}_0\| \leq K_3 (h + c |\zeta_0| \sqrt{h}),$$

for certain $K_3 > 0$. Since $|\zeta_0| \leq A_h = \sqrt{2k |\ln h|}$, and $h |\ln h| \rightarrow 0$ as $h \rightarrow 0$, we can choose sufficiently small h_1 such that for $h < h_1$, $K_3 (h + c |\zeta_0| \sqrt{h}) < 1$, which implies $\boldsymbol{\phi}(z) \in [L_1 - 1, L_2 + 1]^{2m}$.

On the other hand, denote

$$\mathbf{f}(z) := \mathbf{J}^{-1} \nabla_z K(z, \mathbf{C}), \mathbf{g}(z) := c \mathbf{J}^{-1} \nabla_z K(z, \mathbf{C}),$$

then for any $z_1, z_2 \in [L_1 - 1, L_2 + 1]^{2m}$, due to smoothness of K and $\frac{z_i + \mathbf{Z}_0}{2} \in [L_1 - 1, L_2 + 1]^{2m}$ ($i = 1, 2$), it is not difficult to see from (11) that there exists constant $K_4 > 0$ such that

$$\|\mathbf{f}(z_1) - \mathbf{f}(z_2)\| \leq K_4 \|z_1 - z_2\|.$$

Choose $h_2 > 0$ such that for $h < h_2$, $K_4 (h + c |\zeta_0| \sqrt{h}) < 1$, which implies $\|\boldsymbol{\phi}(z_1) - \boldsymbol{\phi}(z_2)\| < \|z_1 - z_2\|$. Then for $h < h_0 := \min\{h_1, h_2\}$, $\boldsymbol{\phi}: [L_1 - 1, L_2 + 1]^{2m} \rightarrow [L_1 - 1, L_2 + 1]^{2m}$ is a contraction mapping. According to the contraction mapping principle, the sequence of iterations converges and $\mathbf{Z}_1 \in [L_1 - 1, L_2 + 1]^{2m}$.

Expanding the midpoint scheme, by the smoothness of K and the boundedness of \mathbf{Z}_0 and \mathbf{Z}_1 , we get

$$\mathbf{Z}_1 = \mathbf{Z}_0 + h \mathbf{f} \left(\frac{\mathbf{Z}_1 + \mathbf{Z}_0}{2} \right) + \mathbf{g} \left(\frac{\mathbf{Z}_1 + \mathbf{Z}_0}{2} \right) \zeta_0 \sqrt{h}$$

$$= \mathbf{Z}_0 + \left(\mathbf{f}(\mathbf{Z}_0) + \frac{1}{2} \frac{\partial \mathbf{f}}{\partial \mathbf{z}}(\mathbf{Z}_0)(\mathbf{Z}_1 - \mathbf{Z}_0) + O(h) \right) h + \left(\mathbf{g}(\mathbf{Z}_0) + \frac{1}{2} \frac{\partial \mathbf{g}}{\partial \mathbf{z}}(\mathbf{Z}_0)(\mathbf{Z}_1 - \mathbf{Z}_0) + \frac{1}{8} \frac{\partial^2 \mathbf{g}}{\partial \mathbf{z}^2}(\mathbf{Z}_0)(\mathbf{Z}_1 - \mathbf{Z}_0)^2 + O(h\sqrt{h}) \right) \zeta_0 \sqrt{h}.$$

Then using the relationship (8), we have

$$\begin{aligned} \mathbf{Z}_1 &= \mathbf{Z}_0 + \mathbf{f}(\mathbf{Z}_0)h + \mathbf{g}(\mathbf{Z}_0)\zeta_0\sqrt{h} + \frac{1}{2} \frac{\partial \mathbf{f}}{\partial \mathbf{z}}(\mathbf{Z}_0) \left(\mathbf{g} \left(\frac{\mathbf{Z}_1 + \mathbf{Z}_0}{2} \right) \zeta_0 \sqrt{h} + \mathbf{f} \left(\frac{\mathbf{Z}_1 + \mathbf{Z}_0}{2} \right) h \right) h + \frac{1}{2} \frac{\partial \mathbf{g}}{\partial \mathbf{z}}(\mathbf{Z}_0) \left(\mathbf{g} \left(\frac{\mathbf{Z}_1 + \mathbf{Z}_0}{2} \right) \zeta_0 \sqrt{h} \right) + \mathbf{f} \left(\frac{\mathbf{Z}_1 + \mathbf{Z}_0}{2} \right) h \zeta_0 \sqrt{h} + \frac{1}{8} \frac{\partial^2 \mathbf{g}}{\partial \mathbf{z}^2}(\mathbf{Z}_0) \left(\mathbf{g} \left(\frac{\mathbf{Z}_1 + \mathbf{Z}_0}{2} \right) \zeta_0 \sqrt{h} + \mathbf{f} \left(\frac{\mathbf{Z}_1 + \mathbf{Z}_0}{2} \right) h \right)^2 \zeta_0 \sqrt{h} + O(h^2) \\ &= \mathbf{Z}_0 + \mathbf{f}(\mathbf{Z}_0)h + \mathbf{g}(\mathbf{Z}_0)\zeta_0\sqrt{h} + \frac{1}{2} \frac{\partial \mathbf{g}}{\partial \mathbf{z}}(\mathbf{Z}_0)\mathbf{g}(\mathbf{Z}_0)\zeta_0^2 h + \frac{1}{4} \left(\frac{\partial \mathbf{g}}{\partial \mathbf{z}}(\mathbf{Z}_0) \right)^2 \mathbf{g}(\mathbf{Z}_0)\zeta_0^3 h \sqrt{h} + \frac{1}{2} \frac{\partial \mathbf{f}}{\partial \mathbf{z}}(\mathbf{Z}_0)\mathbf{g}(\mathbf{Z}_0)\zeta_0 h \sqrt{h} + \frac{1}{2} \frac{\partial \mathbf{g}}{\partial \mathbf{z}}(\mathbf{Z}_0)\mathbf{f}(\mathbf{Z}_0)\zeta_0 h \sqrt{h} + \frac{1}{8} \frac{\partial^2 \mathbf{g}}{\partial \mathbf{z}^2}(\mathbf{Z}_0)\mathbf{g}^2(\mathbf{Z}_0)\zeta_0^3 h \sqrt{h} + O(h^2). \end{aligned}$$

Since

$$\mathbf{Z}(t_1) = \mathbf{Z}_0 + \int_{t_0}^{t_1} \mathbf{f}(\mathbf{Z}(s)) ds + \int_{t_0}^{t_1} \mathbf{g}(\mathbf{Z}(s)) \circ dW(s).$$

Expanding $\mathbf{Z}(s)$ at \mathbf{Z}_0 and inserting the relationship into the above equation, we have

$$\begin{aligned} \mathbf{Z}(t_1) &= \mathbf{Z}_0 + \mathbf{f}(\mathbf{Z}_0)h + \mathbf{g}(\mathbf{Z}_0)\Delta W_0 + \frac{1}{2} \frac{\partial \mathbf{g}}{\partial \mathbf{z}}(\mathbf{Z}_0)\mathbf{g}(\mathbf{Z}_0)(\Delta W_0)^2 + \left(\frac{\partial \mathbf{g}}{\partial \mathbf{z}}(\mathbf{Z}_0) \right)^2 \mathbf{g}(\mathbf{Z}_0) \int_{t_0}^{t_1} \int_{t_0}^s \int_{t_0}^u \circ dW(p) \circ dW(u) \circ dW(s) + \frac{\partial \mathbf{g}}{\partial \mathbf{z}}(\mathbf{Z}_0)\mathbf{f}(\mathbf{Z}_0) \int_{t_0}^{t_1} \int_{t_0}^s du \circ dW(s) + \frac{\partial \mathbf{f}}{\partial \mathbf{z}}(\mathbf{Z}_0)\mathbf{g}(\mathbf{Z}_0) \int_{t_0}^{t_1} \int_{t_0}^s \circ dW(u) ds + \frac{1}{2} \frac{\partial^2 \mathbf{g}}{\partial \mathbf{z}^2}(\mathbf{Z}_0)\mathbf{g}^2(\mathbf{Z}_0) \int_{t_0}^{t_1} \left(\int_{t_0}^s \circ dW(u) \right)^2 \circ dW(s) + O(h^2). \end{aligned}$$

Using the properties of Stratonovich multiple integrals^[12] and the relationship (9), we have

$$\begin{aligned} \|\mathbb{E}(\mathbf{Z}_1 - \mathbf{Z}(t_1))\| &= O(h^2), \\ (\mathbb{E} \|\mathbf{Z}_1 - \mathbf{Z}(t_1)\|^2)^{1/2} &= O(h^{3/2}). \end{aligned} \quad (13)$$

Then by the Chebyshev's inequality, we get

$$\begin{aligned} P(\|\mathbf{Z}_1 - \mathbf{Z}(t_1)\| - \mathbb{E} \|\mathbf{Z}_1 - \mathbf{Z}(t_1)\| \geq h) &\leq P(|\|\mathbf{Z}_1 - \mathbf{Z}(t_1)\| - \mathbb{E} \|\mathbf{Z}_1 - \mathbf{Z}(t_1)\|| \geq h) \\ &\leq \frac{\text{Var} \|\mathbf{Z}_1 - \mathbf{Z}(t_1)\|}{h^2}. \end{aligned}$$

We have

$$\begin{aligned} \text{Var} \|\mathbf{Z}_1 - \mathbf{Z}(t_1)\| &= \mathbb{E} \|\mathbf{Z} - \mathbf{Z}(t_1)\|^2 - (\mathbb{E} \|\mathbf{Z}_1 - \mathbf{Z}(t_1)\|)^2 \\ &\leq O(h^3) + O(h^4) = O(h^3). \end{aligned}$$

Thus

$$P(\|\mathbf{Z}_1 - \mathbf{Z}(t_1)\| \geq \mathbb{E} \|\mathbf{Z}_1 - \mathbf{Z}(t_1)\| + h) \leq O(h).$$

Since

$$\mathbb{E} \|\mathbf{Z}_1 - \mathbf{Z}(t_1)\| \leq (\mathbb{E} \|\mathbf{Z}_1 - \mathbf{Z}(t_1)\|^2)^{1/2} = O(h^{3/2}),$$

we have

$$P(\|\mathbf{Z}_1 - \mathbf{Z}(t_1)\| \geq O(h^{3/2}) + h) \leq O(h).$$

Therefore

$$\begin{aligned} P(\|\mathbf{Z}_1 - \mathbf{Z}(t_1)\| \geq 2h) &\leq P(\|\mathbf{Z}_1 - \mathbf{Z}(t_1)\| \geq O(h^{3/2}) + h) \\ &\leq O(h). \end{aligned}$$

Since $\mathbf{Z}(t_1) \in [L_1, L_2]^{2m}$, then we can derive

$$\begin{aligned} P(\mathbf{Z}_1 \in [L_1 - 2h, L_2 + 2h]^{2m}) &\geq P(\|\mathbf{Z}_1 - \mathbf{Z}(t_1)\| \leq 2h) \\ &\geq 1 - O(h). \end{aligned}$$

Let $h \rightarrow 0$, we obtain $P(\mathbf{Z}_1 \in [L_1, L_2]^{2m}) = 1$. That is, for $\mathbf{Z}_0 \in [L_1, L_2]^{2m}$, we have proved that $\mathbf{Z}_1 \in [L_1, L_2]^{2m}$ almost surely.

Similarly, repeat the process for $\mathbf{Z}_j, j = 1, \dots, N - 1$, we can obtain that $\mathbf{Z}_j \in [L_1, L_2]^{2m}$ almost surely for $j = 2, \dots, N$. By the Darboux-Lie theorem we know that $\boldsymbol{\theta}^{-1}$ exists and is continuous. Therefore $\mathbf{y}_j = \boldsymbol{\theta}^{-1}(\mathbf{Z}_j, \mathbf{C})$ ($j = 0, \dots, N$) are bounded almost surely. □

Remark 2.1 Similar to the discussion in Ref. [5] (Remark 3.5), although the system (7) may not have global Lipschitz coefficients, the stochastic fundamental convergence theorem by Milstein et. al. given in Ref. [12] (Theorem 1.1 of Ref. [12]) can still be applied, i. e., the equation (13) can imply that

$$\mathbb{E} \|\mathbf{Z}_N - \mathbf{Z}(T)\|^2 \leq O(h^2), \quad (14)$$

namely, the midpoint method applied to the system (7) is of root mean-square order 1. This is because

for any given $(\mathbf{Z}_0, \mathbf{C}) = \boldsymbol{\theta}(\mathbf{y}_0)$ with $\mathbf{y}_0 > \mathbf{0}, \mathbf{Z}(t)$ ($t \in [t_0, T]$) and $\{\mathbf{Z}_j\}$ ($j = 0, \dots, N$) are almost surely bounded in a certain $[L_1, L_2]^{2m}$, which is a convex compact subset of \mathbb{R}^{2m} , and the coefficients $\mathbf{J}^{-1}\nabla K$ and $c\mathbf{J}^{-1}\nabla K$ of the system (7) are continuously differentiable functions such that they are Lipschitz continuous on $[L_1, L_2]^{2m}$, which then ensures the validity of the stochastic fundamental convergence theorem on this system (see proof of Theorem 1.1 in Ref. [12]).

Theorem 2.2 Under Assumption 2.1, and assuming the Jacobian matrix of $\boldsymbol{\theta}^{-1}(\mathbf{Z}, \mathbf{C})$ is continuous, then the TM method for system (4)-(5) is of root mean-square order 1.

Proof By the Darboux-Lie theorem (Theorem 1.1), we know $\boldsymbol{\theta}^{-1}$ exists and is differentiable. Under the assumption that $(\boldsymbol{\theta}^{-1})'$ is continuous, and using (14), we have

$$\begin{aligned} \mathbb{E} \|\mathbf{y}_N - \mathbf{y}(T)\|^2 &= \mathbb{E} \|\boldsymbol{\theta}^{-1}(\mathbf{Z}_N, \mathbf{C}) - \boldsymbol{\theta}^{-1}(\mathbf{Z}(T), \mathbf{C})\|^2 \\ &= \mathbb{E} \|(\boldsymbol{\theta}^{-1})'(\boldsymbol{\xi}, \mathbf{C})(\mathbf{Z}_N - \mathbf{Z}(T), \boldsymbol{\theta})\|^2 \\ &\leq K_5 \mathbb{E} \|\mathbf{Z}_N - \mathbf{Z}(T)\|^2 \leq O(h^2), \end{aligned} \tag{15}$$

for certain $\boldsymbol{\xi} = \tau \mathbf{Z}_N + (1 - \tau)\mathbf{Z}(T)$ ($\tau \in [0, 1]$) and $K_5 > 0$ which is the bound of the norm of the continuous matrix-valued function $(\boldsymbol{\theta}^{-1})'$ on the compact subset $[\bar{L}_1, \bar{L}_2]^n$ of \mathbb{R}^n , where $\bar{L}_1 = \min\{L_1, \mathbf{C}_1, \dots, \mathbf{C}_l\}$, $\bar{L}_2 = \max\{L_2, \mathbf{C}_1, \dots, \mathbf{C}_l\}$, and \mathbf{C}_i denotes the i -th element of the vector \mathbf{C} . The equation (15) implies that the TM method is of root mean-square convergence order 1. \square

3 Numerical experiments

Consider a three-dimensional L-V system with Stratonovich white noise perturbation^[5]

$$\begin{aligned} d\mathbf{y}(t) &= \mathbf{B}(\mathbf{y}(t)) \nabla H(\mathbf{y}(t)) (dt + c \circ dW(t)), \\ \mathbf{B}(\mathbf{y}(t)) &= \begin{pmatrix} 0 & v y_a y_b & b v y_a y_c \\ -v y_a y_b & 0 & -y_b y_c \\ -b v y_a y_c & y_b y_c & 0 \end{pmatrix}, \end{aligned} \tag{16}$$

$$\begin{aligned} H(\mathbf{y}) &= a b y_a + y_b + \gamma \ln y_b - a y_c - \mu \ln y_c, \\ \mathbf{y}(0) &= \mathbf{y}_0, \end{aligned}$$

where $\mathbf{y} = (y_a, y_b, y_c)^T, \mathbf{y}_0 = (y_{0,a}, y_{0,b}, y_{0,c})^T$, and

$$\mathbf{B} = \begin{pmatrix} 0 & v & b v \\ -v & 0 & -1 \\ -b v & 1 & 0 \end{pmatrix}.$$

The Casimir function of this system is

$$\begin{aligned} \mathbf{C}(\mathbf{y}) &= -\frac{1}{v} \ln y_a - b \ln y_b + \ln y_c \\ &\equiv -\frac{1}{v} \ln y_{0,a} - b \ln y_{0,b} + \ln y_{0,c} := C. \end{aligned}$$

Let $\mathbf{y}_0 = [1.0, 1.9, 0.5]^T$, and the parameters $a = -2, b = -1, v = -0.5, \gamma = 1, \mu = 2, c = 0.2$. We can check that the vectors $\boldsymbol{\beta} = (ab, 1, -a), \boldsymbol{p} = (0, -\gamma, \mu)$ satisfy the Assumption 2.1 by taking $s_0 = 1, \boldsymbol{\alpha} = (-6, -3, -3)^T$. Then the exact solution of the system (16) is almost surely bounded and positive.

Similar to the discussion in section 3.3 of Ref. [1], based on the Darboux-Lie Theorem 1.1, we can find the following coordinate transformation $\bar{\mathbf{y}} = \boldsymbol{\theta}(\mathbf{y})$ that converts (16) to its canonical form (7):

$$\bar{y}_a = \ln y_c, \bar{y}_b = -\ln y_b, \bar{y}_c = C, \tag{17}$$

by solving the equations:

$$\begin{aligned} \{\bar{y}_a, \bar{y}_a\} &= 0, \{\bar{y}_a, \bar{y}_b\} = -1, \{\bar{y}_a, \bar{y}_c\} = 0, \\ \{\bar{y}_b, \bar{y}_a\} &= 1, \{\bar{y}_b, \bar{y}_b\} = 0, \{\bar{y}_b, \bar{y}_c\} = 0, \\ \{\bar{y}_c, \bar{y}_a\} &= 0, \{\bar{y}_c, \bar{y}_b\} = 0, \{\bar{y}_c, \bar{y}_c\} = 0. \end{aligned} \tag{18}$$

Owing to skew-symmetry of Poisson brackets, the nine equations in (18) can be reduced to the following three equations

$$\{\bar{y}_b, \bar{y}_a\} = 1, \{\bar{y}_c, \bar{y}_a\} = 0, \{\bar{y}_c, \bar{y}_b\} = 0. \tag{19}$$

We can choose \bar{y}^3 to be the Casimir function $\mathbf{C}(\mathbf{y}) \equiv C$ which fulfills the last two equations naturally. The first equation is a partial differential equation with unknown functions $\bar{y}_a = \bar{y}_a(y_a, y_b, y_c), \bar{y}_b = \bar{y}_b(y_a, y_b, y_c)$ and their partial derivatives, which may possess many solutions, implying that the coordinate transformations may not be unique. If we let $\bar{y}_a = \ln y_c$, we find that $\bar{y}_b = -\ln y_b$ solves the equation. Thus we get the coordinate transformation (17). In general, for a given Poisson system satisfying the conditions of the Darboux-Lie Theorem 1.1, one can find such coordinate transformations by solving the partial differential equations in (6) defined by the Poisson bracket associated with the structural matrix

$\mathbf{B}(\mathbf{y})$ of the system. The Casimir functions can be found by exploring the kernel space of $\mathbf{B}(\mathbf{y})$, according to the definition of a Casimir function.

Denoting $(\bar{y}_a, \bar{y}_b) = (P, Q)$, we get the following SHS

$$d \begin{bmatrix} P \\ Q \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \nabla K(P, Q) (dt + c \circ dW(t)), \tag{20}$$

where $K(P, Q) = ab \exp(v(C - P - bQ)) + \exp(-Q) - \gamma Q - a \exp(P) - \mu P$. We apply the midpoint method to the system (20) to get $(P_j, Q_j), j = 0, 1, \dots, N$. Then by using the inverse transformation $\mathbf{y}_j = (\exp(v(C - P_j - bQ_j)), \exp(-Q_j), \exp(P_j))^T$, we obtain the numerical solution $\{\mathbf{y}_j, j = 0, 1, \dots\}$ for the stochastic L-V system (16).

In Fig. 1 we show one sample path of y_a, y_b , and y_c produced by our method. It can be seen that the numerical solution is positive and bounded, coinciding well with the reference exact solution of system (16). Hereby we take $h = 10^{-3}$, and the exact solution is approximated by applying the midpoint method to system (16) with tiny time step 10^{-5} .

Figure 2 shows the evolution of the Casimir function with respect to t produced by our method and the midpoint method directly applied to the system (16). Clearly our method can exactly preserve the Casimir function, while the midpoint

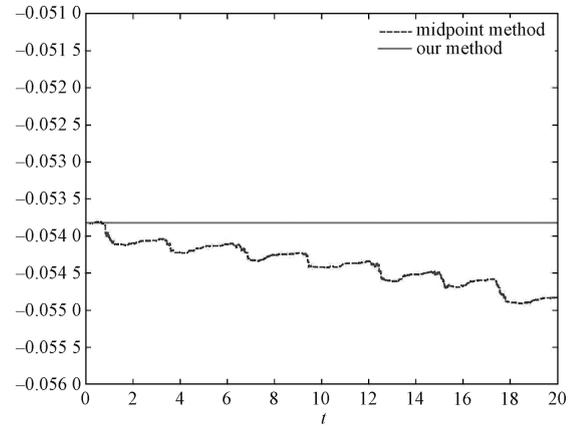


Fig. 2 Casimir evolution by our method and the midpoint method

method fails to preserve the Casimir function. Parameters here are the same with those for Fig. 1.

Figure 3 illustrates the evolution of the Hamiltonian $H(\mathbf{y})$ of the system (16) produced by our method and the midpoint method. It can be seen that our method can nearly preserve the Hamiltonian function of the system, while the midpoint method produces much larger error in the Hamiltonian evolution. Parameters are the same with those for Fig. 1.

Finally, Fig. 4 shows that our method is of root mean-square convergence order 1, coinciding with the theoretical result in Theorem 2.2. Hereby $h = [2^{-11}, 2^{-10}, 2^{-9}, 2^{-8}, 2^{-7}]$, and 500 sample paths are taken to approximate the expectation. Other parameters are the same with those for Fig. 1.

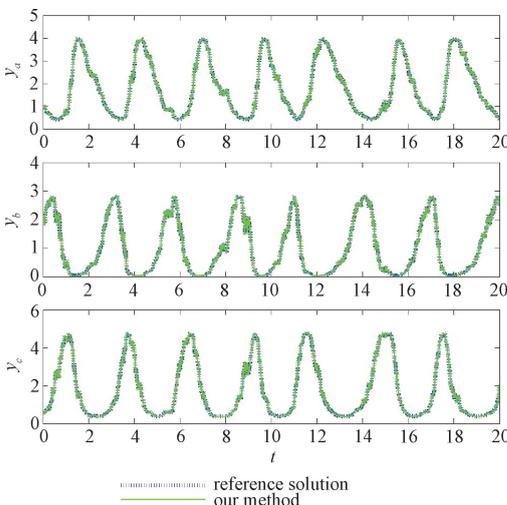


Fig. 1 Sample paths arising from our method

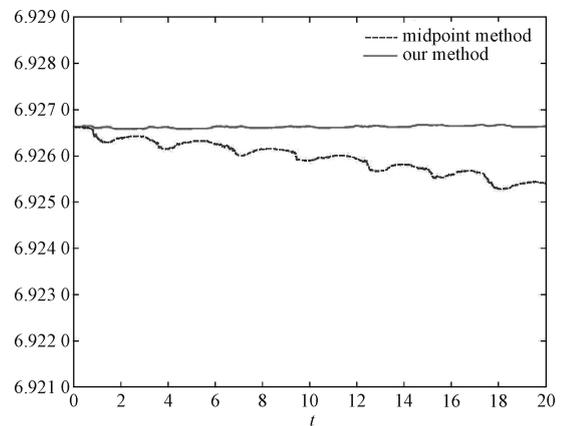


Fig. 3 Hamiltonian evolution by our method and the midpoint method

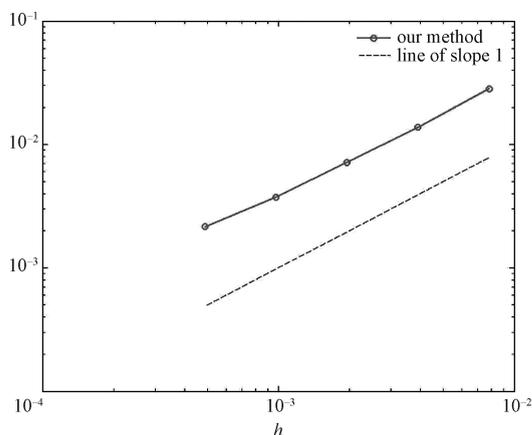


Fig. 4 Root mean-square convergence order

4 Conclusion

We propose a numerical algorithm based on the Darboux-Lie theorem and the midpoint method for a class of stochastic Poisson systems, which can preserve both the Poisson structure and the Casimir functions of the original systems. We also prove that the proposed method has root mean-square convergence order 1. Numerical tests verify the theoretical results and illustrate the effectiveness of the method.

References

- [1] Hong J L, Ruan J L, Sun L Y, et al. Structure-preserving numerical methods for stochastic Poisson systems [J]. *Communications in Computational Physics*, 2021, 29 (3): 802-830. DOI:10.4208/cicp. oa-2019-0084.
- [2] Hairer E, Wanner G, Lubich C. *Geometric numerical integration [M]*. Berlin, Heidelberg: Springer Berlin Heidelberg, 2002. DOI:10.1007/978-3-662-05018-7.
- [3] Arnold V I. *Mathematical methods of classical mechanics [M]*. 2nd ed. New York, NY: Springer New York, 1989. DOI:10.1007/978-1-4757-2063-1.
- [4] Milstein G N, Repin Y M, Tretyakov M V. Symplectic integration of Hamiltonian systems with additive noise [J]. *SIAM Journal on Numerical Analysis*, 2002, 39 (6): 2066-2088. DOI:10.1137/S0036142901387440.
- [5] Cohen D, Dujardin G. Energy-preserving integrators for stochastic Poisson systems [J]. *Communications in Mathematical Sciences*, 2014, 12 (8): 1523-1539. DOI: 10.4310/CMS.2014. v12. n8. a7.
- [6] Li X Y, Ma Q, Ding X H. High-order energy-preserving methods for stochastic Poisson systems [J]. *East Asian Journal on Applied Mathematics*, 2019, 9 (3): 465-484. DOI:10.4208/eajam.290518.310718.
- [7] Hong J L, Ji L H, Wang X, et al. Stochastic K-symplectic integrators for stochastic non-canonical Hamiltonian systems and applications to the Lotka-Volterra model [EB/OL]. 2017; arXiv: 1711.03258 [math. NA]. <https://arxiv.org/abs/1711.03258>.
- [8] Wang Y C, Wang L J. Analysis of the solutions of a class of stochastic Poisson systems [J]. *Journal of University of Chinese Academy of Sciences*, 2020, to appear. DOI: 10.7523/j.ucas.2020.0057.
- [9] Milstein G N, Repin Y M, Tretyakov M V. Numerical methods for stochastic systems preserving symplectic structure [J]. *SIAM Journal on Numerical Analysis*, 2002, 40 (4): 1583-1604. DOI:10.1137/S0036142901395588.
- [10] Deng J, Anton C, Wong Y S. High-order symplectic schemes for stochastic Hamiltonian systems [J]. *Communications in Computational Physics*, 2014, 16 (1): 169-200. DOI: 10.4208/cicp.311012.191113a.
- [11] Ruan J L, Wang L J, Wang P J. Exponential discrete gradient schemes for a class of stochastic differential equations [J]. *Journal of Computational and Applied Mathematics*, 2022, 402: 113797. DOI:10.1016/j.cam.2021.113797.
- [12] Milstein G N. *Numerical integration of stochastic differential equations [M]*. Dordrecht: Springer, 1995. DOI:10.1007/978-94-015-8455-5.