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A class of quasilinear equations with −1 powers*

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Abstract This paper deals with quasilinear elliptic equations of singular growth like $-\Delta u - u\Delta(u^2) = a(x)u^{-1}$. We establish the existence of positive solutions for general $a(x) \in L^p(\Omega)$, $p > 2$, where Ω is a bounded domain in $\mathbb{R}^N$ with $N \geqslant 1$.

Keywords quasilinear singular equation; −1 power; elliptic equation

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一类具有−1 幂指数的拟线性奇异偏微分方程

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摘 要 研究一类具有奇异增长的拟线性椭圆方程 $-\Delta u - u\Delta(u^2) = a(x)u^{-1}$。对于一般的 $a(x) \in L^p(\Omega)$, $p > 2$,证明了该方程正解的存在性,其中 Ω 为 $\mathbb{R}^N$ 中的有界区域且 $N \geqslant 1$。

关键词 拟线性奇异方程;−1 幂指数;椭圆方程

Let Ω be a smooth bounded domain of $\mathbb{R}^N$, $N \geqslant 1$ and let $H_0^1(\Omega)$ be the standard Sobolev space consisting of functions which vanish on the boundary of Ω and whose gradient is in $L^2(\Omega)$. We consider the following quasi-linear singular equation

$$\begin{cases} -\Delta u - u\Delta(u^2) = a(x)\dfrac{1}{u} \text{ in } \Omega, \\ u > 0 \quad \text{in } \Omega,\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{1}$$

where $a(x) > 0$ a.e. in Ω and $a \in L^p(\Omega)$ with $p > 2$. This type of equations is closely related to the standing wave solutions of the following quasilinear Schrodinger equation

$$\begin{aligned} i\partial_t\psi = &-\Delta\psi + V(x)\psi - h(x, |\psi|^2)\psi \\ &- \kappa\Delta[\rho(|\psi|^2)]\rho'(|\psi|^2)\psi, \end{aligned} \tag{2}$$

which has wide applications to physical models, suchas the superfluid film equation in plasma physics when $\rho(s) = s$[1], the self-channelling of a high-power ultra short laser in matter when $\rho(s) = \sqrt{1+s}$[2], fluid mechanics[3], dissipative quantum mechanics, and condensed-matter theory[4]. A lot of results have been obtained in, for example, Refs. [5-7] and references therein. Singular problems have been intensively studied since 1970s because of

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their wide applications: boundary layer phenomenon in fluid mechanics, chemical heterogenous catalysts, glacial advance, etc.; we refer the reader to the books by Agarwal and O'Regan[8], and Hernández and Mancebo[9] for an excellent introduction to the singular boundary theory.

For the quasilinear singular equation involving a singular function f

$$\begin{cases} -\Delta u - u\Delta(u^2) = f(x,u) \text{ in } \Omega, \\ u > 0 \quad \text{in } \Omega,\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{3}$$

there are only a few results available about the existence of solutions by now. In Ref. [10], Marcos do Ó and Moameni considered the existence of radially symmetric solutions when Ω is a ball centered at the origin and the nonlinearity $f(u) = \lambda u^3 - u - u^{-\gamma}$ with $-\gamma \in (-1,0)$. In Ref. [11], Santos, Yang and Zhou studied the case $f(x,u) = \lambda a(x)u^{-\gamma} + b(x)u^{\beta}$ with $-\gamma \in (-1,0)$, $\beta \in (1, 2\cdot 2^* - 1)$ and obtained the existence and multiplicity of solutions. For strongly singular quasilinear equations, in Ref. [12] Alves and Reis used the techniques developed in Ref. [13] by the second author for pure strongly singular equations and established the sufficient and necessary condition for the existence of solutions in the case when $f(x,u) = h(x)u^{-\gamma} + g(x,u)$ with $-\gamma < -1$. When $-\gamma = -1$, Alves[14] and Liu et al.[15] established the existence results for all $-\gamma < 0$ and $f(x,u) = a(x)u^{-\gamma} + \lambda u^p$ with coefficient function $a(x)$ satisfying: for each $-\gamma < 0$ there exists $0 \leqslant \varphi_0 \in C(\overline{\Omega})$ and $p > N$ such that $a(x)\varphi_0^{-\gamma} \in L^p(\Omega)$, which is used to construct the upper-lower solutions to overcome the singularity. In this paper we study the case $-\gamma = -1$ and face it in the quasilinear singular problem (1) for general $a(x) \in L^p(\Omega)$ with $p > 2$.

1 The main result

Theorem 1.1 Let $\Omega \subset \mathbb{R}^N$, $N \geqslant 1$, be a bounded domain with smooth boundary $\partial\Omega$. If $a(x) > 0$ a. e. in Ω and $a \in L^p(\Omega)$ with $p > 2$, then problem (1) admits a positive $H_0^1(\Omega)$-solution.

By solutions of (1) we mean here solutions in $H_0^1(\Omega)$, i. e. $u \in H_0^1(\Omega)$ satisfying $u(x) > 0$ in Ω and for all $\psi \in H_0^1(\Omega)$,

$$\int_\Omega \nabla u \nabla \psi + 2u^2 \nabla u \nabla \psi + 2u\psi |\nabla u|^2 - a(x)\frac{1}{u}\psi \mathrm{d}x = 0.$$

We consider the following functional on $H_0^1(\Omega)$:

$$J(u) = \frac{1}{2}\int_\Omega (1 + 2u^2)|\nabla u|^2 \mathrm{d}x - \int_\Omega a(x)\ln(u)\mathrm{d}x.$$

In this situation, one must find the integrability of both $\int_\Omega |\nabla u|^2 u^2 \mathrm{d}x$ and $\int_\Omega a(x)\ln u \mathrm{d}x$ is obscure on the function space $H_0^1(\Omega)$, and so the functional I is not well defined on the entire space $H_0^1(\Omega)$. In addition, when one faces quasilinear terms, a change of functions relying upon a nonlinear ODE equation: h is the solution of the following problem

$$\begin{cases} h'(t) = \dfrac{1}{\sqrt{1 + 2h^2(t)}}, & t \geqslant 0, \\ h(t) = -h(-t), & t \leqslant 0. \end{cases}$$

(c. f. Ref. [16]) and the change, defined by $\omega(x) = h^{-1}(u(x))$, $\forall x \in \Omega$, make singular terms more delicate. We give a way to deal with the singular functional after the function transform and establish the existence of solutions for more general coefficient functions of singular nonlinearities.

Notation. In the paper we make use of the following notation:

c denotes (possibly different) constants;

We denote by $\|u\|^2 = \int_\Omega |\nabla u|^2 \mathrm{d}x$ the norm in $H_0^1(\Omega)$ and for $u \in H_0^1(\Omega)$, $u^+ = \max\{u, 0\}$.

2 Reformulation of the problem (P)

First, we collect some properties of the function $h(t)$ defined by the solution of the nonlinear equation:

$$\begin{cases} h'(t) = \dfrac{1}{\sqrt{1 + 2h^2(t)}}, & t \geqslant 0, \\ h(t) = -h(-t), & t \leqslant 0. \end{cases}$$

Lemma 2.1 (c. f. Refs. [5, 16-18]) Let h be defined as above. Then h has the following properties:

1) $h''(t) = -2h(t)(h'(t))^4$, $t > 0$;

2) h is unique, invertible and $C^\infty(\mathbb{R})$-function;

3) $0 \leqslant h'(t) \leqslant 1$, $\forall t \in \mathbb{R}$;

4) $|h(t)| \leqslant |t|$ and $|h(t)|^2 \leqslant \sqrt{2}|t|$, $\forall t \in \mathbb{R}$;

5) $\dfrac{h(t)}{2} \leqslant th'(t) \leqslant h(t)$, $t \geqslant 0$;

6) $|h(t)| \geqslant h(1)|t|$, $|t| \leqslant 1$ and $|h(t)| \geqslant h(1)|t|^{\frac{1}{2}}$, $|t|>1$;

7) $\lim\limits_{t\to 0}\dfrac{h(t)}{t}=1$, $\lim\limits_{t\to\infty}\dfrac{h(t)}{t}=0$ and $\lim\limits_{t\to\infty}\dfrac{h(t)}{\sqrt{t}}=2^{\frac{1}{4}}$;

8) $|h(t)|h'(t) \leqslant \dfrac{1}{\sqrt{2}}$, $\forall t \in \mathbb{R}$, $h^{\delta}(t)h'(t)$ is increasing, $t>0$ and $\delta>1$;

9) $h(t)h'(t)/t$, $t>0$ is decreasing, $\lim\limits_{t\to 0}\dfrac{h(t)h'(t)}{t}=1$ and $\lim\limits_{t\to\infty}\dfrac{h(t)h'(t)}{t}=0$;

10) $h(t)^{-\alpha}h^{'(t)}t$ is strictly decreasing on $(0,+\infty)$ with $\alpha \geqslant 2$.

Proof We prove the point 10) of this lemma. The properties 1)-9) have been proved in Refs. [5, 16-18]. It is easily checked, thanks to point 5) of this lemma, that

$$t \geqslant h(1+2h^2)^{\frac{1}{2}}\frac{1}{\alpha} \geqslant h\frac{(1+2h^2)^{\frac{3}{2}}}{\left(1+\left(2+\dfrac{2}{\alpha}\right)h^2\right)}\frac{1}{\alpha},$$

since $\alpha \geqslant 2$. Then it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}[h(t)^{-\alpha}h'(t)t]$$

$$=\frac{h(1+2h^2)^{\frac{3}{2}}-(\alpha+(2\alpha+2)h^2)t}{h^{\alpha+1}(1+2h^2)^2}<0. \quad \square$$

By the change of function $\omega(x)=h^{-1}(u(x))$, we see that if $\omega \in H_0^1(\Omega)$ is a solution of

$$\begin{cases} -\Delta\omega = a(x)\dfrac{1}{h(\omega)}h'(\omega) \text{ in } \Omega, \\ \omega>0 \text{ in } \Omega,\ \omega=0 \text{ on } \partial\Omega, \end{cases} \tag{4}$$

namely,

$$\int_{\Omega}\nabla\omega\nabla\psi - a(x)\frac{h'(\omega)}{h(\omega)}\psi\mathrm{d}x=0,\ \forall\psi \in H_0^1(\Omega),$$

then $h(\omega) \in H_0^1(\Omega)$ is the solution of (1). Now we write that

$$I(\omega)=\frac{1}{2}\int_{\Omega}|\nabla\omega|^2\mathrm{d}x - \int_{\Omega}a(x)\ln h(\omega(x))\mathrm{d}x.$$

Note that, after the transformation of $u(x) \in H_0^1(\Omega)$ into $h^{-1}(u(x))=\omega(x) \in H_0^1(\Omega)$, the difficulty for the integrability of $\int_{\Omega}u^2|\nabla u|^2\mathrm{d}x$ has transferred to how to control the singular term $\ln h(\omega(x))$, $\forall x \in \Omega$, since $h(0)=0$, in which $h(t)$ the solution of an nonlinear O. D. E. can not be expressed. We first restrict our attention to the well defined set of functions in $H_0^1(\Omega)$ for the singular functional $I(\omega)$: X and $\mathcal{N}$ as follows

$$X=\{\omega \in H_0^1(\Omega): \omega>0 \text{ a.e. in } \Omega,\ a(x)\ln\omega(x) \in L^1(\Omega)\},$$

$$\mathcal{N}=\{\omega \in X: \|\omega\|^2-\int_{\Omega}a(x)\frac{h'(\omega)}{h(\omega)}\omega\mathrm{d}x=0\}.$$

Indeed, since $h(0)=0$ and $h(t)$ is strictly increasing on $[0,+\infty)$, thanks to properties 4) and 6) of Lemma 2.1, we get

$$\left|\int_{A_+}a(x)\ln h(\omega(x))\mathrm{d}x\right| \leqslant \int_{A_+}a(x)|\ln h(\omega(x))|\mathrm{d}x$$
$$=\int_{A_+}a(x)\ln h(\omega(x))\mathrm{d}x \leqslant \int_{A_+}a(x)\ln\omega(x)\mathrm{d}x<+\infty,$$

$$\left|\int_{A_-}a(x)\ln h(\omega(x))\mathrm{d}x\right| \leqslant \int_{A_-}a(x)|\ln h(\omega(x))|\mathrm{d}x$$
$$=\int_{A_-}a(x)\ln\left(\frac{1}{h(\omega(x))}\right)\mathrm{d}x \leqslant$$
$$\int_{A_-\cap[\omega<1]}a(x)\left[\ln\left(\frac{1}{h(1)}\right)+\ln\left(\frac{1}{\omega(x)}\right)\right]\mathrm{d}x+$$
$$\int_{A_-\cap[\omega\geqslant 1]}a(x)\left[\ln\left(\frac{1}{h(1)}\right)+\frac{1}{2}\ln\left(\frac{1}{\omega(x)}\right)\right]\mathrm{d}x$$
$$\leqslant 2|\ln h(1)|\int_{\Omega}a(x)\mathrm{d}x+\frac{3}{2}\int_{\Omega}a(x)|\ln\omega(x)|\mathrm{d}x<+\infty,$$

where

$$A_+=\{x \in \Omega: \ln h(\omega(x))>0\},$$
$$A_-=\{x \in \Omega: \ln h(\omega(x)) \leqslant 0\},$$

which clearly implies that the singular functional I is well defined on X. In addition, for any $t>0$ and $\omega \in X$,

$$I(t\omega)=t^2\frac{1}{2}\|\omega\|^2-\int_{\Omega}a(x)\ln h(t\omega)\mathrm{d}x,$$

so that

$$\frac{\mathrm{d}}{\mathrm{d}t}I(t\omega)=t\|\omega\|^2-\int_{\Omega}a(x)\frac{h'(t\omega)}{h(t\omega)}\omega\mathrm{d}x.$$

Note that, thanks to property 5) of Lemma 2.1, we get

$$\frac{h'(t\omega(x))}{h(t\omega(x))}\omega(x) \leqslant (t\omega(x))^{-1}\omega(x)=t^{-1},\ \forall x \in \Omega.$$

Thus $\int_\Omega a(x)\dfrac{h'(t\omega)}{h(t\omega)}\omega\mathrm{d}x$ makes sense.

Claim 2.1 The set X is not empty.

Proof of Claim 2.1 We let φ_1 be the first positive eigenfunction of $-\Delta$ in Ω with Dirichlet boundary condition, that is, $-\Delta\varphi_1=\lambda_1\varphi_1, \varphi_1|_{\partial\Omega}=0$, with λ_1 the first Dirichlet eigenvalue of $-\Delta$. It is well known that

$$\int_\Omega \varphi_1^{-\gamma}(x)\mathrm{d}x<\infty \tag{5}$$

if and only if $-\gamma>-1$ (c.f. Ref. [19]). By the result of Ref. [19] (Theorem 1 and 2), we also know that: for any $-\alpha\in(-3,-1)$, there exist two positive constants d_0, d_1 and $\omega_\alpha(x)\in C^2(\Omega)\cap C(\bar{\Omega})\cap H_0^1(\Omega)$, $\omega_\alpha(x)>0$ in Ω such that

$$d_0\varphi_1^{\frac{2}{1+\alpha}}(x)\leqslant\omega_\alpha(x)\leqslant d_1\varphi_1^{\frac{2}{1+\alpha}}(x), \forall x\in\Omega. \tag{6}$$

We choose $\alpha_0=\dfrac{2p-1}{p-1}$ so that $-\alpha_0\in(-3,-1)$ and $-\dfrac{2p'}{1+\alpha_0}>-1$ since $p>2$, where $p'=\dfrac{p}{p-1}$. Then it is possible to find the corresponding function $\omega_{\alpha_0}(x)\in H_0^1(\Omega)$, $\omega_{\alpha_0}(x)>0, \forall x\in\Omega$ verifying $d_0\varphi_1^{\frac{2}{1+\alpha_0}}(x)\leqslant\omega_{\alpha_0}(x)\leqslant d_1\varphi_1^{\frac{2}{1+\alpha_0}}(x), \forall x\in\Omega$. For such a $\omega_{\alpha_0}\in H_0^1(\Omega)$, $\omega_{\alpha_0}(x)>0\ \forall x\in\Omega$, we divide the domain Ω into three parts:

$$A_{\omega_{\alpha_0}}=\{x\in\Omega:\ln\omega_{\alpha_0}<-1\},$$
$$B_{\omega_{\alpha_0}}=\{x\in\Omega:|\ln\omega_{\alpha_0}|\leqslant 1\},$$
$$D_{\omega_{\alpha_0}}=\{x\in\Omega:\ln\omega_{\alpha_0}>1\},$$

and then we have

$$\left|\int_{B_{\omega_{\alpha_0}}} a(x)\ln\omega_{\alpha_0}\mathrm{d}x\right|\leqslant\int_{B_{\omega_{\alpha_0}}} a(x)|\ln\omega_{\alpha_0}|\mathrm{d}x$$
$$\leqslant\int_{B_{\omega_{\alpha_0}}} a(x)\mathrm{d}x\leqslant\int_\Omega a(x)\mathrm{d}x<+\infty,$$
$$\left|\int_{D_{\omega_{\alpha_0}}} a(x)\ln\omega_{\alpha_0}\mathrm{d}x\right|=\int_{D_{\omega_{\alpha_0}}} a(x)\ln\omega_{\alpha_0}\mathrm{d}x$$
$$\leqslant\int_\Omega a(x)\omega_{\alpha_0}\mathrm{d}x\leqslant c\left(\int_\Omega a(x)^p\mathrm{d}x\right)^{\frac{1}{p}}\|\omega_{\alpha_0}\|<+\infty,$$
$$\left|\int_{A_{\omega_{\alpha_0}}} a(x)\ln\omega_{\alpha_0}\mathrm{d}x\right|\leqslant\int_{A_{\omega_{\alpha_0}}} a(x)|\ln\omega_{\alpha_0}|\mathrm{d}x$$
$$\leqslant\int_{A_{\omega_{\alpha_0}}} a(x)\frac{1}{\omega_{\alpha_0}}\mathrm{d}x\leqslant\int_\Omega a(x)\frac{1}{\omega_{\alpha_0}}\mathrm{d}x<+\infty,$$

where we have used the fact that

$$\int_\Omega a(x)\omega_{\alpha_0}^{-1}\mathrm{d}x\leqslant d_0^{-1}\int_\Omega a(x)\varphi_1^{\frac{2}{1+\alpha_0}}(x)\mathrm{d}x$$
$$\leqslant d_0^{-1}\left(\int_\Omega a^p(x)\mathrm{d}x\right)^{\frac{1}{p}}\left(\int_\Omega\varphi_1^{-\frac{2p'}{1+\alpha_0}}(x)\mathrm{d}x\right)^{\frac{1}{p'}},$$

so that $a(x)\ln\omega_{\alpha_0}(x)\in L^1(\Omega)$ and the set X is nonempty. This ends the proof of Claim 2.1. □

Claim 2.2 For every $\omega\in X$ there exists some $t(\omega)>0$ (which may be not unique) such that $t(\omega)\omega\in\mathcal{N}$ and $I(t(\omega)\omega)\leqslant I(t\omega)$, $\forall t>0$.

Proof of Claim 2.2 Fix $\omega\in X$. We set

$$f(t)=t\|\omega\|^2-\int_\Omega a(x)\frac{1}{h(t\omega)}h'(t\omega)\omega\mathrm{d}x,\ \forall t>0.$$

Thanks to property 5) of Lemma 2.1, we have that

$$\frac{1}{2}\leqslant\frac{h'(t\omega(x))}{h(t\omega(x))}t\omega(x)\leqslant 1,\ \forall x\in\Omega.$$

We set

$$g_{i,\omega}(t)=t\|\omega\|^2-\frac{1}{it}\int_\Omega a(x)\mathrm{d}x,\ \forall t>0,\ i=1,2, \tag{7}$$

so that $f(t)$ verifies

$$g_{1,\omega}(t)\leqslant f(t)\leqslant g_{2,\omega}(t),\ \forall t>0.$$

Clearly, $g_{i,\omega}(t), i=1,2$, is strictly increasing for all $t>0$ and satisfies

$$g_{i,\omega}(t)\to+\infty \text{ as } t\to+\infty,\ g_{i,\omega}(t)\to-\infty \text{ as } t\to0^+,$$

then it follows that

$$f(t)\to+\infty \text{ as } t\to+\infty,\ f(t)\to-\infty \text{ as } t\to0^+.$$

Moreover, since $f(t)=\dfrac{\mathrm{d}}{\mathrm{d}t}I(t\omega), \forall t>0$, $I(t\omega)$ is decreasing on $t>0$ small and increasing on $t>0$ large, so that there must exists some $t(\omega)>0$ (may be not unique) such that $I(t\omega)\geqslant I(t(\omega)\omega)$, $\forall t>0$, which gives

$$f(t(\omega))=\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=t(\omega)}I(t\omega)=0. \tag{8}$$

We thus obtain, thanks to $\omega\in X$ and (8), that $a(x)\ln(t(\omega)\omega(x))\in L^1(\Omega)$, $t(\omega)\omega(x)\in\mathcal{N}$. □

Proof of Theorem 1.1 We know from the property 4) of Lemma 2.1 that there exists $c_X\in\mathbb{R}$ such that for any function $\omega\in X$,

$$I(\omega)=\frac{1}{2}\int_\Omega|\nabla\omega|^2\mathrm{d}x-\int_\Omega a(x)\ln h(\omega(x))\mathrm{d}x$$
$$\geqslant\frac{1}{2}\|\omega\|^2-\int_\Omega a(x)h(\omega(x))\mathrm{d}x$$

$$\geqslant \frac{1}{2}\|\omega\|^2 - \int_\Omega a(x)\omega(x)\,dx \geqslant \frac{1}{2}\|\omega\|^2 - \left(\int_\Omega a(x)^p dx\right)^{\frac{1}{p}}\left(\int_\Omega \omega^{p'}(x)\,dx\right)^{\frac{1}{p'}}$$

$$\geqslant \frac{1}{2}\|\omega\|^2 - c\|\omega\| \geqslant c_X,\ \forall \omega \in X, \tag{9}$$

where we have used the fact that $\ln s \leqslant s, \forall s > 0$. This, with Claim 2.1 implies $\inf_{\forall \omega \in X} I(\omega)$ is a finite number. Claim 2.2 independently gives that

$\inf_{\forall \omega \in X} I(\omega) \leqslant \inf_{\forall \omega \in \mathcal{N}} I(\omega) \leqslant I(t(\omega)\omega) \leqslant I(\omega), \forall \omega \in X.$

Thus, we have obtained

$$\inf_X I = \inf_{\mathcal{N}} I \in \mathbb{R}.$$

We let $\{\omega_n\} \subset X$ be the sequence such that $I(\omega_n) \to \inf_X I$, which gives with $\inf_X I$ a finite number and (8), that $\{\omega_n\} \subset X$ is bounded in $H_0^1(\Omega)$. Then there exist $\omega_0 \in H_0^1(\Omega), g \in L^2(\Omega)$ and a subsequence, still denoted by ω_n, such that

$$\begin{cases} \omega \rightharpoonup \omega_0 \text{ in } H_0^1(\Omega), \\ \omega_n \to \omega_0 \text{ a.e. in } \Omega, \\ \omega_n \to \omega_0 \text{ in } L^2(\Omega), \\ |\omega_n(x)| \leqslant g(x),\ |\omega_0(x)| \leqslant g(x) \text{ a.e. in } \Omega. \end{cases} \tag{10}$$

Since $\{\omega_n\} \subset X$ and $\omega_n(x) > 0$ a.e. $x \in \Omega$, we know $\omega_0(x) \geqslant 0$, a.e. $x \in \Omega$. We prove now $\omega_0(x) > 0$ a.e. $x \in \Omega$ and $a(x)\ln\omega_0(x) \in L^1(\Omega)$ i.e. $\omega_0 \in X$. Since $\{\omega_n\} \subset X$, using (9), we write that

$$\frac{1}{2}\|\omega_n\|^2 - c_X \geqslant \int_\Omega a(x)\omega_n(x)\,dx$$

$$\geqslant \int_\Omega a(x)\ln\omega_n(x)\,dx \geqslant \int_\Omega a(x)\ln h(\omega_n(x))\,dx$$

$$= \frac{1}{2}\|\omega_n\|^2 - I(\omega_n),$$

where we have used property 4) of Lemma 2.1 for h. By the boundedness of $\{\omega_n\}$ in $H_0^1(\Omega)$ and $I(\omega_n) \to \inf_X I$ which is a finite number, we write

$$\int_\Omega a(x)\ln h(\omega_n(x))\,dx = O(1),$$

$$\int_\Omega a(x)\ln\omega_n(x)\,dx = O(1), \tag{11}$$

$$\int_\Omega a(x)\omega_n(x)\,dx = O(1).$$

Taking advantage of $a(x)(\ln\omega_n(x) - \omega_n(x)) \leqslant 0$ a.e. in Ω and using Fatou's Lemma, one also gets $a(x)\limsup_{n\to+\infty}(\ln\omega_n(x) - \omega_n(x))$ is integrable on Ω, and

$$\limsup_{n\to+\infty}\int_\Omega a(x)(\ln\omega_n(x) - \omega_n(x))\,dx$$

$$\leqslant \int_\Omega a(x)\limsup_{n\to+\infty}(\ln\omega_n(x) - \omega_n(x))\,dx, \tag{12}$$

so that, since

$$a(x)\limsup_{n\to+\infty}(\ln\omega_n(x) - \omega_n(x)) = \begin{cases} -\infty, & \omega_0(x) = 0, \\ a(x)[\ln\omega_0(x) - \omega_0(x)], & \omega_0(x) > 0, \end{cases}$$

we thus obtain, thanks to (11) and (12) and $a(x)[\ln\omega_0(x) - \omega_0(x)] \leqslant 0, \forall x \in \Omega$, that

$$\omega_0(x) > 0 \text{ a.e. } x \in \Omega.$$

Since $p > 2$ (so $p' < 2$) and $\omega_n \to \omega_0$ in $L^2(\Omega)$, using Hölder's inequality, we get

$$\lim_{n\to+\infty}\int_\Omega a(x)\omega_n(x)\,dx = \int_\Omega a(x)\omega_0(x)\,dx. \tag{13}$$

We write now, with $\omega_0(x) > 0$ a.e. in Ω, $\ln\omega_n(x) - \omega_n(x) \leqslant 0$ a.e. in Ω, and using Fatou's lemma, that

$$\limsup_{n\to+\infty}\int_\Omega a(x)\ln\omega_n(x)\,dx - \int_\Omega a(x)\omega_0(x)\,dx$$

$$= \limsup_{n\to+\infty}\int_\Omega a(x)\ln\omega_n(x)\,dx - \limsup_{n\to+\infty}\int_\Omega a(x)\omega_n(x)\,dx$$

$$\leqslant \limsup_{n\to+\infty}\int_\Omega a(x)(\ln\omega_n(x) - \omega_n(x))\,dx$$

$$\leqslant \int_\Omega a(x)\limsup_{n\to+\infty}(\ln\omega_n(x) - \omega_n(x))\,dx$$

$$= \int_\Omega a(x)(\ln\omega_0(x) - \omega_0(x))\,dx$$

$$= \int_\Omega a(x)\ln\omega_0(x)\,dx - \int_\Omega a(x)\omega_0(x)\,dx.$$

Since $\int_\Omega a(x)\omega_0(x)\,dx$ is finite, using (11), we have that

$$-\infty < \limsup_{n\to+\infty}\int_\Omega a(x)\ln\omega_n(x)\,dx$$

$$\leqslant \int_\Omega a(x)\ln\omega_0(x)\,dx \leqslant \int_\Omega a(x)\omega_0(x)\,dx < +\infty,$$

so that we obtain $a(x)\ln\omega_0(x) \in L^1(\Omega)$. It is worthy remarking here that the main difference between dealing with the singularities u^{-1} and $u^{-\nu}$ ($\nu \neq 1$) lies in the energy controlling: $\int_\Omega \ln u(x)\,dx$ should be controlled from both sides since $\ln s < 0, \forall s \in (0,1)$, $\ln s \to -\infty$ as $s \to 0^+$ and $\ln s > 0, \forall s > 1$, and $\int_\Omega u^{1-\nu}\mathrm{d}x$ just need to be controlled form one side since $s^{1-\nu}>0, \forall s>0$.

Then we prove that $I(\omega_0)=\inf_X I$. Moreover, $\omega_0\in\mathcal{N}$. Indeed, since

$$a(x)\ln h(\omega_n(x))\leqslant a(x)\ln(\omega_n(x))\leqslant a(x)\omega_n(x)\leqslant a(x)g(x)$$

and $a(x)g(x)\in L^1(\Omega)$, using Fatou's lemma and $\omega_n(x)\to\omega_0$ a. e. in Ω, we get

$$\limsup_{n\to+\infty}\int_\Omega a(x)\ln h(\omega_n(x))\mathrm{d}x\leqslant\int_\Omega a(x)\ln h(\omega_0(x))\mathrm{d}x, \tag{14}$$

so that

$$\begin{aligned}\inf_X I&=\lim_{n\to+\infty}I(\omega_n)=\liminf_{n\to+\infty}\int_\Omega \frac{1}{2}|\nabla\omega_n|^2-a(x)\ln h(\omega_n(x))\mathrm{d}x\geqslant\liminf_{n\to+\infty}\frac{1}{2}\int_\Omega|\nabla\omega_n|^2\mathrm{d}x+\\&\liminf_{n\to+\infty}\int_\Omega -a(x)\ln h(\omega_n(x))\mathrm{d}x\\&\geqslant\frac{1}{2}\|\omega_0\|^2+\int_\Omega -a(x)\ln h(\omega_0(x))\mathrm{d}x=I(\omega_0)\\&\geqslant I(t(\omega_0)\omega_0)\geqslant\inf_{\mathcal{N}}I=\inf_X I,\end{aligned}$$

where we have used the weak lower semicontinuity of norm, (14), and

$$\liminf_{n\to+\infty}\int_\Omega -a(x)\ln h(\omega_n(x))\mathrm{d}x=-\limsup_{n\to+\infty}\int_\Omega a(x)\ln h(\omega_n(x))\mathrm{d}x.$$

This proves that

$$I(\omega_0)=\inf_X I. \tag{15}$$

This leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=1}I(t\omega_0)=0,$$

that is, $\omega_0\in\mathcal{N}$.

We now show that $h(\omega_0)$ is a solution of problem (1). Suppose $\varphi\in H_0^1(\Omega)$, $\varphi(x)\geqslant 0$, $\forall x\in\Omega$ and $\varepsilon>0$. Set $v=\omega_0+\varepsilon\varphi$, we divide the domain Ω into three parts:

$$A_v=\{x\in\Omega:\ln v<-1\},$$
$$B_v=\{x\in\Omega:|\ln v|\leqslant 1\},$$
$$D_v=\{x\in\Omega:\ln v>1\},$$

and then we have

$$\begin{aligned}&\left|\int_{B_v}a(x)\ln(\omega_0+\varepsilon\varphi)\mathrm{d}x\right|\\&\leqslant\int_{B_v}a(x)|\ln(\omega_0+\varepsilon\varphi)|\mathrm{d}x\\&\leqslant\int_{B_v}a(x)\mathrm{d}x\leqslant\int_\Omega a(x)\mathrm{d}x<+\infty,\end{aligned}$$

$$\begin{aligned}&\left|\int_{D_v}a(x)\ln(\omega_0+\varepsilon\varphi)\mathrm{d}x\right|=\int_{D_v}a(x)\ln(\omega_0+\varepsilon\varphi)\mathrm{d}x\\&\leqslant\int_\Omega a(x)(\omega_0+\varepsilon\varphi)\mathrm{d}x\\&\leqslant c\left(\int_\Omega a(x)^p\mathrm{d}x\right)^{\frac{1}{p}}\|\omega_0+\varepsilon\varphi\|<+\infty,\end{aligned}$$

$$\begin{aligned}&\left|\int_{A_v}a(x)\ln(\omega_0+\varepsilon\varphi)\mathrm{d}x\right|\\&\leqslant\int_{A_v}a(x)|\ln(\omega_0+\varepsilon\varphi)|\mathrm{d}x\\&=\int_{A_v}a(x)\ln\frac{1}{(\omega_0+\varepsilon\varphi)}\mathrm{d}x\leqslant\int_{A_v}a(x)\ln\frac{1}{\omega_0}\mathrm{d}x\\&\leqslant\int_{A_v}a(x)|\ln\omega_0|\mathrm{d}x\\&\leqslant\int_\Omega a(x)|\ln\omega_0|\mathrm{d}x<+\infty,\end{aligned}$$

so $\omega_0+\varepsilon\varphi\in X$. By Claim 2.2 and (15) we can conclude that

$$I(\omega_0+\varepsilon\varphi)\geqslant I(t(\omega_0+\varepsilon\varphi)(\omega_0+\varepsilon\varphi))\geqslant\inf_{\mathcal{N}}I=I(\omega_0),$$

and then we have

$$\frac{\|\omega_0+\varepsilon\varphi\|^2-\|\omega_0\|^2}{2}\geqslant\int_\Omega a(x)\ln h(\omega_0+\varepsilon\varphi)\mathrm{d}x-\int_\Omega a(x)\ln h(\omega_0)\mathrm{d}x. \tag{16}$$

Dividing (16) by $\varepsilon>0$ and letting $\varepsilon\to 0^+$ we conclude with Fatou's Lemma that

$$\begin{aligned}&\int_\Omega\nabla\omega_0\nabla\varphi\mathrm{d}x\\&\geqslant\liminf_{\varepsilon\to0^+}\frac{\int_\Omega a(x)[\ln h(\omega_0+\varepsilon\varphi)-\ln h(\omega_0)]\mathrm{d}x}{\varepsilon}\\&\geqslant\int_\Omega\liminf_{\varepsilon\to0^+}\frac{a(x)[\ln h(\omega_0+\varepsilon\varphi)-\ln h(\omega_0)]}{\varepsilon}\mathrm{d}x\\&=\int_\Omega a(x)\frac{h'(\omega_0)}{h(\omega_0)}\varphi\mathrm{d}x.\end{aligned} \tag{17}$$

Suppose $\psi\in H_0^1(\Omega)$ and $t>0$. Inserting $\varphi=(\omega_0+t\psi)^+$ into (17), we get

$$0 \leqslant \frac{1}{t}\int_{\Omega} \nabla\omega_0\nabla(\omega_0 + t\psi)^+ - a(x)\frac{h'(\omega_0)}{h(\omega_0)}(\omega_0 + t\psi)^+ \mathrm{d}x = \frac{1}{t}\left(\int_{\Omega} - \int_{\Omega\cap[\omega_0+t\psi<0]}\right)\left(\nabla\omega_0\nabla(\omega_0 + t\psi) - a(x)\frac{h'(\omega_0)}{h(\omega_0)}(\omega_0 + t\psi)\right)\mathrm{d}x$$
$$\leqslant \int_{\Omega}\nabla\omega_0\nabla\psi - a(x)\frac{h'(\omega_0)}{h(\omega_0)}\psi\mathrm{d}x - \int_{\Omega\cap[\omega_0+t\psi<0]}\nabla\omega_0\nabla\psi\mathrm{d}x. \tag{18}$$

Since meas $[\omega_0 + t\psi] \to 0$ as $t \to 0^+$, by passing to the limit as $t \to 0^+$ we can conclude that

$$0 \leqslant \int_{\Omega}\nabla\omega_0\nabla\psi - a(x)\frac{h'(\omega_0)}{h(\omega_0)}\psi\mathrm{d}x.$$

The same conclusion can be drawn for $-\psi$ in place of ψ, thus it follows that $h(\omega_0)$ is indeed a $H_0^1(\Omega)$-solution of (1). □

References

[1] Kurihara S. Large-amplitude quasi-solitons in superfluid films [J]. Journal of the Physical Society of Japan, 1981, 50(10):3262-3267. DOI: 10.1143/jpsj.50.3262.

[2] Hartmann B, Zakrzewski W J. Electrons on hexagonal lattices and applications to nanotubes [J]. Physical Review B, 2003, 68(18):184302. DOI: 10.1103/physrevb.68.184302.

[3] Kosevich A M, Ivanov B A, Kovalev A S. Magnetic solitons [J]. Physics Reports, 1990, 194(3/4):117-238. DOI: 10.1016/0370-1573(90)90130-T.

[4] Makhankov V G, Fedyanin V K. Non-linear effects in quasi-one-dimensional models of condensed matter theory [J]. Physics Reports, 1984, 104(1): 1-86. DOI: 10.1016/0370-1573(84)90106-6.

[5] Liu J Q, Wang Y Q, Wang Z Q. Soliton solutions for quasilinear Schrödinger equations, Ⅱ [J]. Journal of Differential Equations, 2003, 187(2): 473-493. DOI: 10.1016/s0022-0396(02)00064-5.

[6] Fang X D, Szulkin A. Multiple solutions for a quasilinear Schrödinger equation [J]. Physics Reports, 2013, 254(4): 2015-2032. DOI: 10.1016/j.jde.2012.11.017.

[7] Liu X Q, Liu J Q, Wang Z Q. Quasilinear elliptic equations via perturbation method [J]. Proceedings of the American Mathematical Society, 2013, 141(1): 253-263. DOI: 10.1090/s0002-9939-2012-11293-6.

[8] Agarwal R P, O'Regan D. Singular differential and integral equations with applications [M]. Dordretht, Netherlands: Springer Science & Business Media, 2003. DOI: 10.1007/978-94-017-3004-4.

[9] Hernández J, Mancebo F J. Singular elliptic and parabolic equations [M] // Handbook of differential equations: stationary partial differential equations, Amsterdam; Boston: Elsevier/North Holland. 2006, 3:317-400. DOI: 10.1016/s1874-5733(06)80008-2.

[10] Marcos do Ó J, Moameni A. Solutions for singular quasilinear Schrödinger equations with one parameter [J]. Communications on Pure & Applied Analysis, 2010, 9(4): 1011-1023. DOI: 10.3934/cpaa.2010.9.1011.

[11] Santos C A, Yang M B, Zhou J Z. Global multiplicity of solutions for a modified elliptic problem with singular terms [J]. Nonlinearity, 2021, 34(11): 7842-7871. DOI: 10.1088/1361-6544/ac2a50.

[12] Alves R L, Reis M. About existence and regularity of positive solutions for a quasilinear Schrödinger equation with singular nonlinearity [J]. Electronic Journal of Qualitative Theory of Differential Equations, 2020, 60: 1-23. DOI: 10.14232/ejqtde.2020.1.60.

[13] Sun Y J. Compatibility phenomena in singular problems [J]. Proceedings of the Royal Society of Edinburgh Section A: Mathematics, 2013, 143(6):1321-1330. DOI: 10.1017/s030821051100117x.

[14] Alves R L. Existence, nonexistence and multiplicity of positive solutions for singular quasilinear problems [J]. Electronic Journal of Qualitative Theory of Differential Equations, 2022, 13:1-29. DOI: 10.14232/ejqtde.2022.1.13.

[15] Liu J Y, Liu D C, Zhao P H. Soliton solutions for a singular Schrödinger equation with any growth exponents [J]. Acta Applicandae Mathematicae, 2017, 148(1):179-199. DOI: 10.1007/s10440-016-0084-z.

[16] do Ó J M B, Miyagaki O H, Soares S H M. Soliton solutions for quasilinear Schrödinger equations: the critical exponential case [J]. Nonlinear Analysis: Theory, Methods & Applications, 2007, 67(12):3357-3372. DOI: 10.1016/j.na.2006.10.018.

[17] do Ó J M B, Miyagaki O H, Soares S H M. Soliton solutions for quasilinear Schrödinger equations with critical growth [J]. Journal of Differential Equations, 2010, 248(4):722-744. DOI: 10.1016/j.jde.2009.11.030.

[18] Colin M, Jeanjean L. Solutions for a quasilinear Schrödinger equation: a dual approach [J]. Nonlinear Analysis: Theory, Methods & Applications, 2004, 56(2): 213-226. DOI: 10.1016/j.na.2003.09.008.

[19] Lazer A C, McKenna P J. On a singular nonlinear elliptic boundary-value problem [J]. Proceedings of the American Mathematical Society, 1991, 111(3):721. DOI: 10.1090/s0002-9939-1991-1037213-9.