

L^p -Bounds of a class of mean operators on product spaces^{*}

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Abstract We characterize a sufficient and necessary condition which ensures that the operator

$$\mathcal{H}_\varphi f(x) = \int_0^1 \cdots \int_0^1 f(x_1 t_1, \cdots, x_n t_n) \varphi(t_1, \cdots, t_n) dt_1 \cdots dt_n$$

is bounded on $L^p(\mathbb{G}^n)$ with $1 \leq p \leq \infty$. The condition deeply depends on the nonnegative function φ defined on $[0, 1] \times \cdots \times [0, 1]$. Furthermore, the corresponding operator norms are worked out.

Key words Hardy operator; product space; L^p

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乘积空间上的一类平均算子的 L^p 有界性

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摘 要 主要研究乘积空间上的一类算子

$$\mathcal{H}_\varphi f(x) = \int_0^1 \cdots \int_0^1 f(x_1 t_1, \cdots, x_n t_n) \varphi(t_1, \cdots, t_n) dt_1 \cdots dt_n$$

在 L^p 上有界的充分必要条件, 这个条件完全依赖于定义在 $[0, 1] \times \cdots \times [0, 1]$ 上的非负函数. 此外, 还给出了 \mathcal{H}_φ 的算子范数.

关键词 哈代算子; 乘积空间; L^p

Let f be a non-negative integrable function on $\mathbb{G} = (0, \infty)$. The classical Hardy operator is defined by

$$Hf(x) := \frac{1}{x} \int_0^x f(t) dt, \quad (1)$$

for $x \in \mathbb{G}$.

The following Theorem A due to Hardy^[1] is well-known.

Theorem A If f is a non-negative measurable function on \mathbb{G} and $1 < p < \infty$, then

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$$\|Hf\|_{L^p(G)} \leq \frac{p}{p-1} \|f\|_{L^p(G)}$$

holds, and the constant $\frac{p}{p-1}$ is sharp.

For the case of n -dimensional product space, the classical Hardy operator is defined by

$$\mathcal{H}f(x) := \frac{1}{x_1 \cdots x_n} \int_0^{x_1} \cdots \int_0^{x_n} f(t_1, \cdots, t_n) dt_1 \cdots dt_n, \quad (2)$$

for $x = (x_1, x_2, \cdots, x_n) \in \mathbb{G}^n$, where f is a non-negative measurable function on \mathbb{G}^n .

In 1992, Pachpatte^[2] investigated the operator defined by (2) and obtained the following Theorem B.

Theorem B If f is a non-negative measurable function on \mathbb{G}^n and $1 < p < \infty$, then

$$\|\mathcal{H}f\|_{L^p(\mathbb{G}^n)} \leq \left(\frac{p}{p-1}\right)^n \|f\|_{L^p(\mathbb{G}^n)}, \quad (3)$$

where the constant $\left(\frac{p}{p-1}\right)^n$ in (3) is the best.

For the case of power weight for the operator H , Theorem C was obtained in Ref. [3].

Theorem C Suppose that f is any non-negative measurable function on \mathbb{G}^n , $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$, $p - 1 = (p - 1, p - 1, \cdots, p - 1)$, and $1 < p < \infty$. If $\alpha < p - 1$, then the following inequality

$$\|\mathcal{H}f\|_{L_{x^\alpha}^p(\mathbb{G}^n)} \leq \left(\prod_{i=1}^n \frac{p}{(p - \alpha_i - 1)}\right) \|f\|_{L_{x^\alpha}^p(\mathbb{G}^n)} \quad (4)$$

holds and the constant $\prod_{i=1}^n \frac{p}{(p - \alpha_i - 1)}$ in (4) is the best possible, where $\alpha < p - 1$ means each $\alpha_i < p - 1$ and $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$.

Recently Xiao^[4] considered a kind of new mean operators defined by

$$\mathcal{H}_\psi f(x) = \int_0^1 f(xt) \psi(t) dt, \quad (5)$$

where $\psi: [0, 1] \rightarrow [0, \infty)$ is a function. Evidently the operator H_ψ deeply depends on the nonnegative function ψ . For example, when $n = 1$ and $\psi(x) = 1$ for $x \in [0, 1]$, a simple deduction implies that H_ψ is just reduced to the classical Hardy operator defined by (1). Consequently, H_ψ is the more extensive Hardy operator. It should be pointed out

that the operator defined by (5) is sometimes called the weighted Hardy-Littlewood average operator.

For the operator H_ψ , Theorem D was obtained in Ref. [4].

Theorem D Suppose that $\psi: [0, 1] \rightarrow [0, \infty)$ is a nonnegative function and $p \in [1, \infty)$. Then H_ψ is bounded on $L^p(\mathbb{R}^n)$ if and only if

$$\int_0^1 t^{-\frac{1}{p}} \psi(t) dt < \infty. \quad (6)$$

Moreover, if the inequality (6) holds, then the operator norm of \mathcal{H}_ψ on $L^p(\mathbb{R}^n)$ is given by

$$\|H_\psi\|_{L^p(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n)} = \int_0^1 t^{-\frac{1}{p}} \psi(t) dt. \quad (7)$$

Theorem D characterizes a sufficient and necessary condition which ensures that the operator H_ψ is bounded on $L^p(\mathbb{R}^n)$ with $1 \leq p \leq \infty$. The condition completely depends on the non-negative function ψ defined on $[0, 1]$. Generally speaking, the problem of the boundedness on the product space is usually difficult. The reason is that every variable has its independent scalar. In this paper, we want to extend the operator H_ψ to multi-dimensional product space and investigate the following operator:

$$\mathcal{H}_\varphi f(x) = \int_0^1 \cdots \int_0^1 f(x_1 t_1, \cdots, x_n t_n) \times \varphi(t_1, \cdots, t_n) dt_1 \cdots dt_n, \quad (8)$$

where $\varphi: [0, 1] \times \cdots \times [0, 1] \rightarrow [0, \infty)$ is a function and f is a measurable complex-valued function on \mathbb{G}^n . Without loss of generality, we may assume that f is a non-negative measurable function on \mathbb{G}^n .

In the same way, for the operator \mathcal{H}_φ , when $\varphi(x) = 1$ for $x \in [0, 1] \times \cdots \times [0, 1]$, we have that \mathcal{H}_φ is just reduced to the classical Hardy operator defined by (2) on the product space. Thus, systematic investigation of the L^p -boundedness of \mathcal{H}_φ is significant in theory.

Now let us formulate the dual operator of \mathcal{H}_φ and denote it by \mathcal{H}_φ^* .

It follows from Fubini theorem that

$$\int_{\mathbb{G}^n} H_\varphi f(x) g(x) dx$$

$$\begin{aligned}
&= \int_{\mathbb{G}^n} \left(\int_0^1 \cdots \int_0^1 f(x_1 t_1, \cdots, x_n t_n) \varphi(t) dt \right) g(x) dx \\
&= \int_0^1 \cdots \int_0^1 \left(\int_{\mathbb{G}^n} f(x_1 t_1, \cdots, x_n t_n) g(x) dx \right) \varphi(t) dt \\
&= \int_0^1 \cdots \int_0^1 (t_1 \cdots t_n)^{-1} \int_{\mathbb{G}^n} f(x) g(x_1/t_1, \cdots, x_n/t_n) dx \varphi(t) dt \\
&= \int_{\mathbb{R}^d} f(x) \int_0^1 \cdots \int_0^1 g(x_1/t_1, \cdots, x_n/t_n) (t_1 \cdots t_n)^{-1} \varphi(t) dt dx \\
&= \int_{\mathbb{R}^d} f(x) H_\varphi^* g(x) dx.
\end{aligned}$$

Thus, we have

$$\mathcal{H}_\varphi^* g(x) = \int_0^1 \cdots \int_0^1 \frac{g(x_1/t_1, \cdots, x_n/t_n) \varphi(t)}{t_1 \cdots t_n} dt. \quad (9)$$

The purpose of this paper is to sharpen and extend the main result in Ref. [4] by Xiao.

1 Main results

Now we formulate our main theorems.

Theorem 1.1 Let $\varphi: [0, 1] \times \cdots \times [0, 1] \rightarrow [0, \infty)$ be a nonnegative function and let $1 \leq p \leq \infty$. Then the operator \mathcal{H}_φ defined by (8) is bounded on $L^p(\mathbb{G}^n)$ if and only if

$$\int_0^1 \cdots \int_0^1 (t_1 \cdots t_n)^{-\frac{1}{p}} \varphi(t_1, \cdots, t_n) dt < \infty. \quad (10)$$

Proof Since the case $p = \infty$ is trivial, it suffices to consider the case $1 \leq p < \infty$.

We first prove the sufficiency of Theorem 1.1. Assume (10) holds. It implies from the generalized Minkowski inequality that

$$\begin{aligned}
&\| \mathcal{H}_\varphi f \|_{L^p(\mathbb{G}^n)} \\
&\leq \int_0^1 \cdots \int_0^1 \left(\int_{\mathbb{G}^n} |f(t_1 x_1, \cdots, t_n x_n)|^p dx_1 \cdots dx_n \right)^{\frac{1}{p}} \\
&\quad \varphi(t_1, \cdots, t_n) dt. \quad (11)
\end{aligned}$$

Making the change of variables $y_1 = x_1 t_1, y_2 = x_2 t_2, \cdots, y_n = x_n t_n$ in inequality (11), we have

$$\begin{aligned}
&\| \mathcal{H}_\varphi f \|_{L^p(\mathbb{G}^n)} \leq \| f \|_{L^p(\mathbb{G}^n)} \\
&\int_0^1 \cdots \int_0^1 (t_1 \cdots t_n)^{-\frac{1}{p}} \varphi(t_1, \cdots, t_n) dt. \quad (12)
\end{aligned}$$

Since the inequality (10) holds, this immediately implies that the operator \mathcal{H}_φ is bounded on $L^p(\mathbb{G}^n)$, and

$$\| \mathcal{H}_\varphi \|_{L^p(\mathbb{G}^n)} \leq \int_0^1 \cdots \int_0^1 (t_1 \cdots t_n)^{-\frac{1}{p}} \varphi(t_1, \cdots, t_n) dt.$$

This completes the proof of the sufficiency of Theorem 1.1.

Next we prove the necessity of Theorem 1.1. The proof of the converse comes from the standard integral calculation. If \mathcal{H}_φ is bounded on $L^p(\mathbb{G}^n)$, then there exists a constant $C > 0$ such that

$$\| \mathcal{H}_\varphi f \|_{L^p(\mathbb{G}^n)} \leq C \| f \|_{L^p(\mathbb{G}^n)}. \quad (13)$$

Now, for any $\varepsilon > 0$, set

$$f_\varepsilon^i(x_i) = \begin{cases} 0, & 0 < x_i \leq 1, \\ x_i^{-\frac{1}{p-\varepsilon}}, & x_i > 1, \end{cases}$$

and

$$f_\varepsilon(x_1, x_2, \cdots, x_n) = \prod_{i=1}^n f_\varepsilon^i(x_i). \quad (14)$$

Then, a straightforward computation leads to

$$\| f_\varepsilon \|_{L^p(\mathbb{G}^n)}^p = \left(\frac{p-\varepsilon}{\varepsilon} \right)^n. \quad (15)$$

Obviously, $\mathcal{H}_\varphi f_\varepsilon(\cdots, x_i, \cdots) = 0$ while $0 < x_i$.

Moreover, we also have that, if $x_i > 1$,

$$\begin{aligned}
&\mathcal{H}_\varphi f_\varepsilon(\cdots, x_i, \cdots) \\
&= x_i^{-\frac{1}{p-\varepsilon}} \int_0^1 \cdots \int_0^1 t_i^{-\frac{1}{p-\varepsilon}} \varphi(\cdots t_i \cdots) dt_i. \quad (16)
\end{aligned}$$

Since inequality (13) applies to f_ε , putting $\delta = \varepsilon^{-1} > 1$, we conclude from equation (15) that

$$\begin{aligned}
&C^p \| f_\varepsilon \|_{L^p(\mathbb{G}^n)}^p \\
&\geq \| H_\varphi f_\varepsilon \|_{L^p(\mathbb{G}^n)}^p \\
&= \int_1^\infty \cdots \int_1^\infty (x_1^{-\frac{1}{p-\varepsilon}} \cdots x_n^{-\frac{1}{p-\varepsilon}})^p \\
&\quad \left(\int_{\frac{1}{x_1}}^1 \cdots \int_{\frac{1}{x_n}}^1 t_1^{-\frac{1}{p-\varepsilon}} \cdots t_n^{-\frac{1}{p-\varepsilon}} \varphi(t) dt \right)^p dx \\
&\geq \int_\delta^\infty \cdots \int_\delta^\infty (x_1^{-\frac{1}{p-\varepsilon}} \cdots x_n^{-\frac{1}{p-\varepsilon}})^p \\
&\quad \left(\int_{\frac{1}{\delta}}^1 \cdots \int_{\frac{1}{\delta}}^1 t_1^{-\frac{1}{p-\varepsilon}} \cdots t_n^{-\frac{1}{p-\varepsilon}} \varphi(t) dt \right)^p dx \\
&= \delta^{-\frac{ne}{p-\varepsilon}} \| f_\varepsilon \|_{L^p(\mathbb{G}^n)}^p \\
&\quad \left(\int_{\frac{1}{\delta}}^1 \cdots \int_{\frac{1}{\delta}}^1 t_1^{-\frac{1}{p-\varepsilon}} \cdots t_n^{-\frac{1}{p-\varepsilon}} \varphi(t) dt \right)^p. \quad (17)
\end{aligned}$$

This implies

$$\int_\varepsilon^1 \cdots \int_\varepsilon^1 (t_1 \cdots t_n)^{-\frac{1}{p-\varepsilon}} \varphi(t) dt \leq \frac{C}{\varepsilon^{\frac{ne}{p(p-\varepsilon)}}}. \quad (18)$$

Letting $\varepsilon \rightarrow 0$ in inequality (18) and using the

elementary knowledge of limit in mathematical analysis, we immediately have the inequality (10). \square

Theorem 1.2 When inequality (10) holds, the operator norm of \mathcal{H}_φ on $L^p(\mathbb{G}^n)$ is given by

$$\|\mathcal{H}_\varphi\|_{L^p(\mathbb{G}^n) \rightarrow L^p(\mathbb{G}^n)} = \int_0^1 \cdots \int_0^1 (t_1 t_2 \cdots t_n)^{-\frac{1}{p}} \times \varphi(t_1, \cdots, t_n) dt. \quad (19)$$

Proof When (10) is true, that is, H_φ is bounded on $L^p(\mathbb{G}^n)$, by (12) in Theorem 1.1, the following inequality

$$\|\mathcal{H}_\varphi\|_{L^p(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n)} \leq \int_0^1 \cdots \int_0^1 (t_1 t_2 \cdots t_n)^{-\frac{1}{p}} \times \varphi(t_1, \cdots, t_n) dt \quad (20)$$

must hold.

In order to deduce (19), we assume otherwise that (19) is not true. Then by (20) there is a positive number κ such that

$$\|\mathcal{H}_\varphi\|_{L^p(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n)} \leq \kappa < \int_0^1 \cdots \int_0^1 (t_1 t_2 \cdots t_n)^{-\frac{1}{p}} \varphi(t_1, \cdots, t_n) dt. \quad (21)$$

We further use the above function f_ε in (14). It follows from (21) and (17) that

$$\begin{aligned} & \kappa^p \|\mathcal{H}_\varepsilon\|_{L^p(\mathbb{R}^n)}^p \\ & \geq \|\mathcal{H}_\varphi\|_{L^p(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n)}^p \|\mathcal{H}_\varepsilon\|_{L^p(\mathbb{R}^n)}^p \\ & \geq \|\mathcal{H}_\varphi \mathcal{H}_\varepsilon\|_{L^p(\mathbb{R}^n)}^p \\ & \geq \|\mathcal{H}_\varepsilon\|_{L^p(\mathbb{R}^n)}^p (\delta^{-\frac{n\varepsilon}{p-\varepsilon}})^p \\ & \left(\int_\varepsilon^1 \cdots \int_\varepsilon^1 (t_1 t_2 \cdots t_n)^{-\frac{1}{p-\varepsilon}} \varphi(t_1, \cdots, t_n) dt \right)^p. \end{aligned}$$

This implies

$$\kappa \geq \int_0^1 \cdots \int_0^1 (t_1 t_2 \cdots t_n)^{-\frac{1}{p}} \varphi(t_1, \cdots, t_n) dt. \quad (22)$$

However, the inequality (22) contradicts with the inequality (21). Therefore, the equation (19) must hold. \square

Since the operator \mathcal{H}_φ^* is the dual operator of \mathcal{H}_φ , we can immediately deduce the following results.

Theorem 1.3 Let $\varphi: [0,1] \times \cdots \times [0,1] \rightarrow [0,\infty)$ be a function and let $p \in [1,\infty)$. Then \mathcal{H}_φ^* is bounded on $L^p(\mathbb{G}^n)$ if and only if

$$\int_0^1 \cdots \int_0^1 (t_1 t_2 \cdots t_n)^{-\frac{p-1}{p}} \varphi(t_1, \cdots, t_n) dt < \infty. \quad (23)$$

Theorem 1.4 When inequality (23) holds, the operator norm of \mathcal{H}_φ^* on $L^p(\mathbb{G}^n)$ is given by

$$\begin{aligned} & \|\mathcal{H}_\varphi^*\|_{L^p(\mathbb{G}^n) \rightarrow L^p(\mathbb{G}^n)} \\ & = \int_0^1 \cdots \int_0^1 (t_1 t_2 \cdots t_n)^{-\frac{p-1}{p}} \varphi(t_1, \cdots, t_n) dt. \end{aligned} \quad (24)$$

Remarks The boundedness of the operator \mathcal{H}_φ^* results from the boundedness of the operator \mathcal{H}_φ . Using the almost similar methods in Theorem 1.1 and Theorem 1.2, we can easily have the proofs of Theorem 1.3 and Theorem 1.4. So we omit them. In particular, if $\varphi = 1$, we have from (10) that \mathcal{H}_φ is not bounded on $L^1(\mathbb{G}^n)$. This is just why p must satisfy the condition $p > 1$ in Theorem B. In the same way, when $\varphi = 1$, we can also deduce that \mathcal{H}_φ^* is not bounded on $L^\infty(\mathbb{G}^n)$.

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