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On a parametric Hilbert-type integral inequality and its applications^{*}

LIU Qiong[†]

(Department of Science and Information, Shaoyang University, Shaoyang 422000, Hunan, China)

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Abstract By introducing some parameters and by using the way of weight function and the techniques of real analysis and Laplace's integral transform, a parametric Hilbert-type integral inequality and its equivalent form are given, and their constant factors are proved to be the best values. As applications, some meaningful results are obtained by selecting the special values for the parameters.

Key words Hilbert-type integral inequality; weight function; Laplace's integral transform; the best constant factor

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一个参量化 Hilbert 型积分不等式及其应用

刘琼

(邵阳学院理学与信息科学系, 湖南 邵阳 422000)

摘 要 通过引入多个参数,应用权函数方法、实分析技巧和拉普拉斯积分变换,给出一个参量化 Hilbert 型积分不等式及其等价式,证明它们的常数因子是最佳的. 作为应用,通过选取一些特殊参数值,获得了一些有意义的结果.

关键词 Hilbert 型积分不等式; 权函数; 拉普拉斯积分变换; 最佳常数因子

If $f, g \geqslant 0, f, g \in L^2(0, \infty), 0 < \|f\|_2 := \left\{ \int_0^\infty f^2(x) dx \right\}^{\frac{1}{2}} < \infty, 0 < \|g\|_2 < \infty$, then we have the famous Hilbert integral inequality^[1]:

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dx dy < \pi \|f\|_2 \|g\|_2, \quad (1)$$

where the constant factor π is the best value. In 1925, Hardy-Riesz gave a best extension of (1) by introducing a pair of conjugate index (p, q) as^[2]:

If $p > 1, \frac{1}{p} + \frac{1}{q} = 1, f, g \geqslant 0, 0 < \|f\|_p := \left\{ \int_0^\infty f^p(x) dx \right\}^{\frac{1}{p}} < \infty, 0 < \|g\|_q < \infty$, then

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[†]E-mail:liuqiongxx13@163.com

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dx dy < \frac{\pi}{\sin(\frac{\pi}{p})} \|f\|_p \|g\|_q, \quad (2)$$

where the constant factor $\frac{\pi}{\sin(\frac{\pi}{p})}$ is the best value.

Inequality (2) is called Hardy-Hilbert's integral inequality, which is important in analysis and applications^[1,3]. In recent years, inequality (2) had been improved and extended by Refs. [4-8]. Recently, a Hilbert-type integral inequalities with the best constant factor is obtained^[9]

$$\int_0^\infty \int_0^\infty e^{-xy} f(x) g(y) dx dy < \sqrt{\pi} \|f\|_2 \|g\|_2, \quad (3)$$

where the constant factors $\sqrt{\pi}$ is the best value.

In this paper, by using the way of weight function and the technique of real analysis and Laplace's integral transform, a parametric Hilbert-type integral inequality with the kernel

$$k_{\alpha,\beta}(x,y) := e^{-\alpha xy} \sinh(\beta xy) \quad (\alpha > \beta > 0)$$

is given.

1 Some lemmas

Lemma 1.1 If $p > 1, \frac{1}{p} + \frac{1}{q} = 1, \alpha > \beta > 0, \lambda \geq 0$, we define weight function as

$$\omega(\alpha, \beta, \lambda, x) := \int_0^\infty e^{-\alpha xy} [\sinh(\beta xy)]$$

$$\frac{y^{\frac{\lambda}{2}-1}}{x^{\frac{p(\frac{\lambda}{2}-1)}{q}}} dy, \quad x \in (0, +\infty),$$

$$\omega(\alpha, \beta, \lambda, y) := \int_0^\infty e^{-\alpha xy} [\sinh(\beta xy)]$$

$$\frac{x^{\frac{\lambda}{2}-1}}{y^{\frac{q(\frac{\lambda}{2}-1)}{p}}} dx, \quad y \in (0, +\infty),$$

then we have

$$\omega(\alpha, \beta, \lambda, x) = C(\alpha, \beta, \lambda) x^{(1-\frac{\lambda}{2})p-1},$$

$$\omega(\alpha, \beta, \lambda, y) = C(\alpha, \beta, \lambda) y^{(1-\frac{\lambda}{2})q-1},$$

where

$$C(\alpha, \beta, \lambda) = \int_0^\infty e^{-\alpha u} [\sinh(\beta u)] u^{\frac{\lambda}{2}-1} du$$

$$= \mathcal{L}_\alpha[(\sinh(\beta u)) u^{\frac{\lambda}{2}-1}]. \quad (4)$$

Proof Set $xy = u$. Then we have

$$\begin{aligned} \omega(\alpha, \beta, \lambda, x) &= \int_0^\infty e^{-\alpha xy} [\sinh(\beta xy)] \frac{y^{\frac{\lambda}{2}-1}}{x^{\frac{p(\frac{\lambda}{2}-1)}{q}}} dy \\ &= x^{(1-\frac{\lambda}{2})p-1} \int_0^\infty e^{-\alpha u} [\sinh(\beta u)] u^{\frac{\lambda}{2}-1} du. \end{aligned} \quad (5)$$

Since $\alpha > \beta > 0, \lambda \geq 0$, thus it is easy to verify that the function $[\sinh(\beta u)] u^{\frac{\lambda}{2}-1}$ meet the existence condition of Laplace's integral transformation^[10], and Laplace's integral transformation of $[\sinh(\beta u)] u^{\frac{\lambda}{2}-1}$ is set as $\mathcal{L}_\alpha[(\sinh(\beta u)) u^{\frac{\lambda}{2}-1}]$, it is

$$\begin{aligned} C(\alpha, \beta, \lambda) &= \mathcal{L}_\alpha[(\sinh(\beta u)) u^{\frac{\lambda}{2}-1}] \\ &= \int_0^\infty e^{-\alpha u} [\sinh(\beta u)] u^{\frac{\lambda}{2}-1} du. \end{aligned}$$

Put the above type in (5). We obtain that $\omega(\alpha, \beta, \lambda, x) = C(\alpha, \beta, \lambda) x^{(1-\frac{\lambda}{2})p-1}$. Similarly, we have $\omega(\alpha, \beta, \lambda, y) = C(\alpha, \beta, \lambda) y^{(1-\frac{\lambda}{2})q-1}$. \square

Lemma 1.2 If $p > 1, \frac{1}{p} + \frac{1}{q} = 1, \alpha > \beta > 0, \lambda \geq 0, 0 < \varepsilon < \min\{q\beta, p\beta\}$, and ε small enough, let us define the real functions as follows:

$$\begin{aligned} \tilde{f}(x) &:= \begin{cases} 0, & x \in (0, 1) \\ x^{\frac{p(\frac{\lambda}{2}-1)-\varepsilon}{p}}, & x \in [1, \infty) \end{cases}, \\ \tilde{g}(y) &:= \begin{cases} 0, & y \in (1, \infty) \\ y^{\frac{q(\frac{\lambda}{2}-1)+\varepsilon}{q}}, & y \in (0, 1] \end{cases}, \end{aligned}$$

then we have

$$\begin{aligned} \tilde{J}\mathcal{E} &= \left[\int_0^\infty x^{(1-\frac{\lambda}{2})p-1} \tilde{f}^p(x) dx \right]^{\frac{1}{p}} \times \\ &\quad \left[\int_0^\infty y^{(1-\frac{\lambda}{2})q-1} \tilde{g}^q(y) dy \right]^{\frac{1}{q}} \mathcal{E} = 1, \end{aligned} \quad (6)$$

$$\begin{aligned} \tilde{I}\mathcal{E} &= \mathcal{E} \int_0^\infty \int_0^\infty e^{-\alpha xy} [\sinh(\beta xy)] \tilde{f}(x) \tilde{g}(y) dx dy \\ &> C(\alpha, \beta, \lambda) (1 - o(1)) (\mathcal{E} \rightarrow 0^+). \end{aligned} \quad (7)$$

Proof We easily obtain

$$\begin{aligned} \tilde{J}\mathcal{E} &= \left[\int_0^\infty x^{(1-\frac{\lambda}{2})p-1} \tilde{f}^p(x) dx \right]^{\frac{1}{p}} \times \\ &\quad \left[\int_0^\infty y^{(1-\frac{\lambda}{2})q-1} \tilde{g}^q(y) dy \right]^{\frac{1}{q}} \mathcal{E} \\ &= \left[\int_1^\infty x^{-(1+\varepsilon)} dx \right]^{\frac{1}{p}} \left[\int_0^1 y^{-1+\varepsilon} dy \right]^{\frac{1}{q}} \mathcal{E} = 1. \end{aligned}$$

Since $H(u) = u^{\frac{\lambda}{2}+1} e^{-\alpha u} \sinh(\beta u)$ is continuous in $(0, \infty)$, $\lim_{u \rightarrow 0^+} H(u) = 0$, and $\lim_{u \rightarrow +\infty} H(u) = 0$, there exists $M > 0$, satisfying $H(u) \leq M$. By Fubini's theorem^[11], we have

$$\begin{aligned} \tilde{I}\varepsilon &= \varepsilon \int_0^\infty \int_0^\infty e^{-\alpha xy} [\sinh(\beta xy)] \tilde{f}(x) \tilde{g}(y) dx dy \\ &= \varepsilon \int_1^\infty x^{\frac{p(\frac{\lambda}{2}-1)-\varepsilon}{p}} dx \times \left[\int_0^1 e^{-\alpha xy} [\sinh(\beta xy)] y^{\frac{q(\frac{\lambda}{2}-1)+\varepsilon}{q}} dy \right] \\ &= \varepsilon \int_1^\infty x^{-1-\varepsilon} dx \left[\int_0^x e^{-\alpha u} [\sinh(\beta u)] u^{\frac{\lambda}{2}-1+\frac{\varepsilon}{q}} du \right] \\ &= \varepsilon \int_0^\infty e^{-\alpha u} [\sinh(\beta u)] u^{\frac{\lambda}{2}-1+\frac{\varepsilon}{q}} du - \\ &\quad \varepsilon \int_1^\infty x^{-1-\varepsilon} dx \int_x^\infty e^{-\alpha u} [\sinh(\beta u)] u^{\frac{\lambda}{2}-1+\frac{\varepsilon}{q}} du \\ &> C(\alpha, \beta, \lambda) + o_1(1) - M\varepsilon \int_1^\infty x^{-1} dx \int_x^\infty u^{-2+\frac{\varepsilon}{q}} du \\ &= C(\alpha, \beta, \lambda) + o_1(1) - \frac{M\varepsilon}{(1 - \frac{\varepsilon}{q})^2} \\ &= C(\alpha, \beta, \lambda) (1 - o(1)) (\varepsilon \rightarrow 0^+). \quad \square \end{aligned}$$

2 Main results and applications

If $\theta(x) (> 0)$ is measurable function, $\rho \geq 1$, the function spaces is set as

$$\begin{aligned} L_\theta^\rho(0, \infty) \\ := \left\{ h; \|h\|_{\rho, \theta} := \left\{ \int_0^\infty \theta(x) |h(x)|^\rho dx \right\}^{\frac{1}{\rho}} < \infty \right\}. \end{aligned}$$

Theorem 2.1 If $p > 1, \frac{1}{p} + \frac{1}{q} = 1$, and $\alpha > \beta$

$$\begin{aligned} > 0, \lambda \geq 0, \varphi(x) = x^{(1-\frac{\lambda}{2})p-1}, \psi(y) = y^{(1-\frac{\lambda}{2})q-1}, f, \\ g \geq 0, f \in L_\varphi^p(0, \infty), g \in L_\psi^q(0, \infty), \|f\|_{p, \varphi}, \\ \|g\|_{q, \psi} > 0, \text{ then we have} \end{aligned}$$

$$\begin{aligned} \int_0^\infty \int_0^\infty e^{-\alpha xy} [\sinh(\beta xy)] f(x) g(y) dx dy \\ < C(\alpha, \beta, \lambda) \|f\|_{p, \varphi} \|g\|_{q, \psi}, \quad (8) \end{aligned}$$

where the constant factor $C(\alpha, \beta, \lambda)$ ($C(\alpha, \beta, \lambda)$ is the same as (4)) is the best value.

Proof By Hölder's inequality^[12], Fubini's theorem, and Lemma 1.1, we obtain

$$\begin{aligned} I := \int_0^\infty \int_0^\infty e^{-\alpha xy} [\sinh(\beta xy)] f(x) g(y) dx dy \\ = \int_0^\infty \int_0^\infty e^{-\alpha xy} [\sinh(\beta xy)] f(x) g(y) \times \left[\frac{y^{\frac{\lambda}{2}-1}}{x^{\frac{\lambda}{2}-1}} \right] \left[\frac{x^{\frac{\lambda}{2}-1}}{y^{\frac{\lambda}{2}-1}} \right] dx dy \end{aligned}$$

$$\begin{aligned} &\leq \left[\int_0^\infty \int_0^\infty e^{-\alpha xy} [\sinh(\beta xy)] f^p(x) \frac{y^{\frac{\lambda}{2}-1}}{x^{\frac{p(\frac{\lambda}{2}-1)}{q}}} dx dy \right]^{\frac{1}{p}} \times \\ &\quad \left[\int_0^\infty \int_0^\infty e^{-\alpha xy} [\sinh(\beta xy)] g^q(y) \frac{x^{\frac{\lambda}{2}-1}}{y^{\frac{q(\frac{\lambda}{2}-1)}{p}}} dx dy \right]^{\frac{1}{q}} \\ &= \left\{ \int_0^\infty \omega(\alpha, \beta, \lambda, x) f^p(x) dx \right\}^{\frac{1}{p}} \times \\ &\quad \left\{ \int_0^\infty \omega(\alpha, \beta, \lambda, y) g^q(y) dy \right\}^{\frac{1}{q}} \\ &= C(\alpha, \beta, \lambda) \|f\|_{p, \varphi} \|g\|_{q, \psi}. \quad (9) \end{aligned}$$

If inequality (9) keeps the form of an equality, then according to Ref. [12] there exist two constants A and B , such that they are not all zero and

$$A \frac{y^{\frac{\lambda}{2}-1}}{x^{\frac{p(\frac{\lambda}{2}-1)}{q}}} f^p(x) = B \frac{x^{\frac{\lambda}{2}-1}}{y^{\frac{q(\frac{\lambda}{2}-1)}{p}}} g^q(y) \text{ a. e.}$$

in $(0, \infty) \times (0, \infty)$. It follows that $Ax^{p(1-\frac{\lambda}{2})} f^p(x) = By^{q(1-\frac{\lambda}{2})} g^q(y)$ a. e. on $(0, \infty) \times (0, \infty)$. Assuming that $A \neq 0$, there exists $y > 0$, such that $x^{p(1-\frac{\lambda}{2})-1} f^p(x) = [By^{q(1-\frac{\lambda}{2})} g^q(y)] \frac{1}{Ax}$ a. e. in $x \in (0, \infty)$,

which contradicts the fact that $0 < \|f\|_{p, \varphi} < \infty$. Then inequality (9) keeps the strict form.

If the constant factor $C(\alpha, \beta, \lambda)$ of (8) is not the best value, then there exists a positive $K < C(\alpha, \beta, \lambda)$, such that inequality (8) is still valid if we replace $C(\alpha, \beta, \lambda)$ by K , then by (6) and (7), we have

$$C(\alpha, \beta, \lambda) (1 - o(1)) < K.$$

Let $\varepsilon \rightarrow 0^+$. We obtain $K \geq C(\alpha, \beta, \lambda)$, which contradicts the fact that $K < C(\alpha, \beta, \lambda)$, thus the constant factor $C(\alpha, \beta, \lambda)$ of (8) is the best value. The theorem is proved. \square

Theorem 2.2 If $p > 1, \frac{1}{p} + \frac{1}{q} = 1, \alpha > \beta > 0$,

$$\begin{aligned} \lambda \geq 0, \varphi(x) = x^{p(1-\frac{\lambda}{2})-1}, f \geq 0, f \in L_\varphi^p(0, \infty), \\ \|f\|_{p, \varphi} > 0, \text{ then we have} \end{aligned}$$

$$\begin{aligned} \int_0^\infty y^{\frac{q(\frac{\lambda}{2}-1)+1}{q-1}} dy \left[\int_0^\infty e^{-\alpha xy} [\sinh(\beta xy)] f(x) dx \right]^p \\ < C^p(\alpha, \beta, \lambda) \|f\|_{p, \varphi}^p, \quad (10) \end{aligned}$$

where the constant factor $C^p(\alpha, \beta, \lambda)$ is the best value, and inequality (10) is equivalent to inequality (8).

Proof Set a bounded measurable function as

$$[f(x)]_n := \min\{n, f(x)\} = \begin{cases} f(x), & \text{for } f(x) < n \\ n, & \text{for } f(x) \geq n \end{cases}.$$

Since $0 < \|f\|_{p,\varphi} < \infty$, there exists $n_0 \in \mathbf{N}$, such

that $0 < \int_{\frac{1}{n}}^n \varphi(x) [f(x)]_n^p dx < \infty (n \geq n_0)$, setting

$$g_n(y) := y^{\frac{q(\frac{\lambda}{2}-1)+1}{q-1}} \left[\int_{\frac{1}{n}}^n e^{-\alpha xy} [\sinh(\beta xy)] [f(x)]_n dx \right]^{\frac{p}{q}}$$

$\left(\frac{1}{n} < y < n, n \geq n_0\right)$, when $n \geq n_0$, by (8) we

have

$$\begin{aligned} 0 &< \int_{\frac{1}{n}}^n \psi(y) g_n^q(y) dy = \int_{\frac{1}{n}}^n y^{\frac{q(\frac{\lambda}{2}-1)+1}{q-1}} \times \\ &\left[\int_{\frac{1}{n}}^n e^{-\alpha xy} [\sinh(\beta xy)] [f(x)]_n dx \right]^p dy \\ &= \int_{\frac{1}{n}}^n \int_{\frac{1}{n}}^n e^{-\alpha xy} [\sinh(\beta xy)] [f(x)]_n g_n(y) dx dy \\ &< C(\alpha, \beta, \lambda) \left\{ \int_{\frac{1}{n}}^n \varphi(x) [f(x)]_n^p dx \right\}^{\frac{1}{p}} \times \\ &\left\{ \int_{\frac{1}{n}}^n \psi(y) g_n^q(y) dy \right\}^{\frac{1}{q}}, \end{aligned} \quad (11)$$

$$\begin{aligned} 0 &< \int_{\frac{1}{n}}^n \psi(y) g_n^q(y) dy < C^p(\alpha, \beta, \lambda) \int_0^\infty \varphi(x) f^p(x) dx \\ &= C^p(\alpha, \beta, \lambda) \|f\|_{p,\varphi}^p < \infty. \end{aligned} \quad (12)$$

It follows $0 < \|f\|_{p,\varphi} < \infty$. For $n \rightarrow \infty$, by (8), both (11) and (12) still keep the form of strict inequalities, hence we have ifnequality (10).

On the other hand, by Hölder's inequality, we find

$$\begin{aligned} I &= \int_0^\infty \int_0^\infty e^{-\alpha xy} [\sinh(\beta xy)] f(x) g(y) dx dy \\ &= \int_0^\infty \left[y^{\frac{q(\frac{\lambda}{2}-1)+1}{p(q-1)}} \int_0^\infty e^{-\alpha xy} [\sinh(\beta xy)] f(x) dx \right] \times \\ &\left[y^{\frac{q(1-\frac{\lambda}{2})-1}{p(q-1)}} g(y) \right] dy \\ &< C(\alpha, \beta, \lambda) \|f\|_{p,\varphi} \|g\|_{q,\psi}. \end{aligned}$$

The inequality is (8), which is equivalent to (10).

If the constant factor of (10) is not the best value, thus, by (10), we can obtain a contradiction that the constant factor in (8) is not the best value.

Thus the constant factor $C^p(\alpha, \beta, \lambda)$ in (10) is the best value. \square

Giving $\lambda = 2$ in (8) and (10), by (4), and checking Laplace's integral transform table, we have

$$\begin{aligned} C(\alpha, \beta) &= \int_0^\infty e^{-\alpha u} \sinh(\beta u) du \\ &= \mathcal{L}_\alpha[\sinh \beta u] = \frac{\beta}{\alpha^2 - \beta^2}. \end{aligned}$$

Thus, we obtain the following corollary.

Corollary 2.1 If $p > 1, \frac{1}{p} + \frac{1}{q} = 1, \alpha > \beta > 0$,

$\varphi(x) = x^{-1}$, and $f, g \geq 0, f \in L_\varphi^p(0, \infty), g \in L_\varphi^q(0, \infty)$, $\|f\|_{p,\varphi}, \|g\|_{q,\varphi} > 0$, then we have the following equivalent inequalities

$$\begin{aligned} &\int_0^\infty \int_0^\infty e^{-\alpha xy} [\sinh(\beta xy)] f(x) g(y) dx dy \\ &< \frac{\beta}{\alpha^2 - \beta^2} \|f\|_{p,\varphi} \|g\|_{q,\varphi}, \end{aligned} \quad (13)$$

$$\begin{aligned} &\int_0^\infty y^{\frac{1}{q}-1} \left[\int_0^\infty e^{-\alpha xy} [\sinh(\beta xy)] f(x) dx \right]^p dy \\ &< \left[\frac{\beta}{\alpha^2 - \beta^2} \right]^p \|f\|_{p,\varphi}^p, \end{aligned} \quad (14)$$

where the constant factors $\frac{\beta}{\alpha^2 - \beta^2}$ and $\left[\frac{\beta}{\alpha^2 - \beta^2} \right]^p$ are the best values.

Giving $\lambda = 4$ in (8) and (10), by (4), and using the differential property of Laplace's integral transform, we have

$$\begin{aligned} C(\alpha, \beta) &= \int_0^\infty e^{-\alpha u} u \sinh \beta u du \\ &= \mathcal{L}_\alpha[u \sinh \beta u] = \frac{2\alpha\beta}{(\alpha^2 - \beta^2)^2}. \end{aligned}$$

Thus, we obtain the following corollary.

Corollary 2.2 If $p > 1, \frac{1}{p} + \frac{1}{q} = 1, \alpha > \beta > 0$,

$\varphi(x) = x^{-p-1}, \psi(y) = y^{-q-1}$, and $f, g \geq 0, f \in L_\varphi^p(0, \infty), g \in L_\psi^q(0, \infty)$, $\|f\|_{p,\varphi}, \|g\|_{q,\psi} > 0$, then we have the following equivalent inequalities:

$$\begin{aligned} &\int_0^\infty \int_0^\infty e^{-\alpha xy} [\sinh(\beta xy)] f(x) g(y) dx dy \\ &< \frac{2\alpha\beta}{(\alpha^2 - \beta^2)^2} \|f\|_{p,\varphi} \|g\|_{q,\psi}, \end{aligned} \quad (15)$$

$$\begin{aligned} &\int_0^\infty y^{\frac{q+1}{q}-1} \left[\int_0^\infty e^{-\alpha xy} [\sinh(\beta xy)] f(x) dx \right]^p dy \\ &< \left[\frac{2\alpha\beta}{(\alpha^2 - \beta^2)^2} \right]^p \|f\|_{p,\varphi}^p, \end{aligned} \quad (16)$$

where the constant factors $\frac{2\alpha\beta}{(\alpha^2 - \beta^2)^2}$ and

$\left[\frac{2\alpha\beta}{(\alpha^2-\beta^2)^2}\right]^p$ are the best values.

Giving $\lambda = 0$ in (8) and (10), by (4), and using the integral property of Laplace's integral transform, we have

$$\begin{aligned} C(\alpha,\beta) &= \int_0^\infty e^{-\alpha u} \frac{1}{u} \sinh \beta u du \\ &= \mathcal{L}_\alpha \left[\frac{1}{u} \sinh \beta u \right] \\ &= \ln \sqrt{\frac{\alpha+\beta}{\alpha-\beta}} = \operatorname{arth} \frac{\beta}{\alpha}. \end{aligned}$$

Thus, we obtain the following corollary.

Corollary 2.3 If $p > 1, \frac{1}{p} + \frac{1}{q} = 1, \alpha > \beta > 0,$
 $\varphi(x) = x^{p-1}, \psi(y) = y^{q-1},$ and $f, g \geq 0, f \in L^p_\varphi(0, \infty), g \in L^q_\psi(0, \infty), \|f\|_{p,\varphi}, \|g\|_{q,\psi} > 0,$
then we have the following equivalent inequalities

$$\begin{aligned} &\int_0^\infty \int_0^\infty e^{-\alpha xy} [\sinh(\beta xy)] f(x) g(y) dx dy \\ &< \left[\ln \sqrt{\frac{\alpha+\beta}{\alpha-\beta}} \right] \|f\|_{p,\varphi} \|g\|_{q,\psi}, \end{aligned} \tag{17}$$

$$\begin{aligned} &\int_0^\infty y^{-1} \left[\int_0^\infty e^{-\alpha xy} [\sinh(\beta xy)] f(x) dx \right]^p dy \\ &< \left[\ln \sqrt{\frac{\alpha+\beta}{\alpha-\beta}} \right]^p \|f\|_{p,\varphi}^p, \end{aligned} \tag{18}$$

where the constant factors, $\ln \sqrt{\frac{\alpha+\beta}{\alpha-\beta}}$ and

$\left[\ln \sqrt{\frac{\alpha+\beta}{\alpha-\beta}} \right]^p,$ are the best values.

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