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Uncertainty equalities and uncertainty relation in weak measurement*

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Abstract Uncertainty principle is one of the fundamental principles in quantum mechanics. In this work, we derive two uncertainty equalities, which hold for all pairs of incompatible observables. We also obtain an uncertainty relation in weak measurement which captures the limitations on the preparation of pre- and post-selected ensemble and holds for two non-Hermitian operators corresponding to two non-commuting observables.

Key words uncertainty equality; uncertainty relation; weak value; variance

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不确定等式和弱测量中的不确定关系

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摘 要 不确定原理是量子力学的基本原理之一. 我们导出 2 个不确定等式,适用于 2 个不相容的可观测量. 同时得到一个弱测量中的不确定关系,适用于 2 个不相容可观测量对应的非厄米算符. 它指出了制备前选择和后选择系综的极限.

关键词 不确定等式;不确定关系;弱值;方差

Uncertainty principle is one of the basic tenets of quantum mechanics. The initial spirit of uncertainty principle was postulated by Heisenberg [1]. Kennard[2] first mathematically derived the Heisenberg uncertainty relation . The most famous and popular form is the Heisenberg-Robertson uncertainty relation [3]

$$\Delta A^2 \Delta B^2 \geqslant \left| \frac{1}{2} \langle \psi | [A,B] | \psi \rangle \right|^2, \quad (1)$$

for any observables A and B and any state $|\psi\rangle$, where the variance of an observable X in state $|\psi\rangle$ is defined as $\Delta X^2 = \langle \psi | X^2 | \psi \rangle - \langle \psi | X | \psi \rangle^2$ and the commutator is defined as $[A,B] = AB - BA$. A stronger extension of the Heisenberg-Robertson uncertainty relation (1) was made by Schrödinger[4], which is generally formulated as

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$$\Delta A^2\Delta B^2 \geqslant \left|\frac{1}{2}\langle[A,B]\rangle\right|^2 + \left|\frac{1}{2}\langle\{A,B\}\rangle - \langle A\rangle\langle B\rangle\right|^2, \qquad (2)$$

where the anticommutator is defined as $\{A,B\} = AB + BA$, and $\langle X\rangle$ is defined as the expectation value $\langle\psi | X | \psi\rangle$ for any operator X with respect to the normalized state $|\psi\rangle$.

However, the above two uncertainty relations have the problem that they may be trivial even when A and B are incompatible on the state $|\psi\rangle$. In order to correct this problem, Maccone and Pati [5] presented two stronger uncertainty relations based on the sum of variances. The first uncertainty relation reads

$$\Delta A^2 + \Delta B^2 \geqslant \pm i\langle[A,B]\rangle + |\langle\psi | A \pm iB | \psi^\perp\rangle|^2, \qquad (3)$$

which is valid for arbitrary states $|\psi^\perp\rangle$ orthogonal to the state of the system $|\psi\rangle$, where the sign should be chosen so that $\pm i\langle[A,B]\rangle$ (a real quantity) is positive. The second uncertainty relation is

$$\Delta A^2 + \Delta B^2 \geqslant \frac{1}{2}|\langle\psi_{A+B}^\perp | A + B | \psi\rangle|^2. \qquad (4)$$

Here $|\psi_{A+B}^\perp\rangle \propto (A + B - \langle A + B\rangle)|\psi\rangle$ is a state orthogonal to $|\psi\rangle$. Maccone and Pati also derived an amended Heisenberg-Robertson uncertainty relation

$$\Delta A\Delta B \geqslant \frac{\pm i\frac{1}{2}\langle[A,B]\rangle}{1 - \frac{1}{2}|\langle\psi | \frac{A}{\Delta A} \pm i\frac{B}{\Delta B} | \psi^\perp\rangle|^2}, \qquad (5)$$

which is stronger than the Heisenberg-Robertson uncertainty relation (1).

Recently, two stronger Schrödinger-like uncertainty relations[6] have been proved which go beyond the Maccone and Pati's uncertainty relation. The new relations provide stronger bounds whenever the observables are incompatible on the state $|\psi\rangle$. The first uncertainty relation is

$$\Delta A^2 + \Delta B^2 \geqslant |\langle[A,B]\rangle + \langle\{A,B\}\rangle - 2\langle A\rangle\langle B\rangle| + |<\psi | A - e^{i\alpha}B | \psi^\perp>|^2, \qquad (6)$$

which is valid for arbitrary states $|\psi^\perp\rangle$ orthogonal to the state of the system $|\psi\rangle$ and stronger than the Maccone and Pati's uncertainty relation (3). In (6), α is a real constant. If $\langle\{A,B\}\rangle - 2\langle A\rangle\langle B\rangle > 0$, then $\alpha = \arctan\frac{-i\langle[A,B]\rangle}{\langle\{A,B\}\rangle - 2\langle A\rangle\langle B\rangle}$. If $\langle\{A,B\}\rangle - 2\langle A\rangle\langle B\rangle < 0$, then $\alpha = \pi + \arctan\frac{-i\langle[A,B]\rangle}{\langle\{A,B\}\rangle - 2\langle A\rangle\langle B\rangle}$. While $\langle\{A,B\}\rangle - 2\langle A\rangle\langle B\rangle = 0$, the relation (6) reduces to (3). The second uncertainty relation is

$$\Delta A^2\Delta B^2 \geqslant \frac{\left|\frac{1}{2}\langle[A,B]\rangle\right|^2 + \left|\frac{1}{2}\langle\{A,B\}\rangle - \langle A\rangle\langle B\rangle\right|^2}{\left(1 - \frac{1}{2}|\langle\psi | \frac{A}{\Delta A} - e^{i\alpha}\frac{B}{\Delta B} | \psi^\perp\rangle|^2\right)^2} \qquad (7)$$

which is stronger than the Schrödinger uncertainty relation (2).

These new state-dependent uncertainty relations have some problem[7], but some state-independent uncertainty relations[8-9] are immune from the drawback. Maccone and Pati's uncertainty relations[5] are still very important and have some generalizations. Two variance-based uncertainty equalities were proved recently by Yao et al.[10] on the trend of stronger uncertainty relations[5], for all pairs of incompatible observables A and B. Meanwhile, two uncertainty relations in weak measurement were derived by Pati and Wu[11] for variances of two non-Hermitian operators corresponding to two noncommuting observables.

In this work we derive and prove two uncertainty equalities, which hold for all pairs of incompatible observables A and B. We also give an uncertainty relation in weak measurement for two non-Hermitian operators corresponding to two non-commuting observables.

1 Uncertainty equalities

In this section, we construct and prove two uncertainty equalities, which imply the uncertainty inequalities (6) and (7).

Uncertainty relation 1.

$$\Delta A^2 + \Delta B^2 = |\langle[A,B]\rangle + \langle\{A,B\}\rangle - 2\langle A\rangle\langle B\rangle|$$

$$+\sum_{n=1}^{d-1} |\langle\psi| A - e^{i\alpha}B |\psi_n^\perp\rangle|^2, \quad (8)$$

where $\{|\psi\rangle, |\psi_n^\perp\rangle_{n=1}^{d-1}\}$ comprise an orthonormal complete basis in the d -dimensional Hilbert space.

Proof To prove our uncertainty relation, let us define the operators $\Pi = I - |\psi\rangle\langle\psi|$, $\bar{A} = A - \langle A\rangle I$, and $\bar{B} = B - \langle B\rangle I$ and the state $|\varphi\rangle = (\bar{A} - e^{i\tau}\bar{B})|\psi\rangle$. We have

$$\begin{aligned}\langle\phi|\Pi|\phi\rangle &= \langle\psi|(\bar{A} - e^{-i\tau}\bar{B})|(I - |\psi\rangle\langle\psi|)|(\bar{A} - e^{i\tau}\bar{B})|\psi\rangle \\ &= \langle\psi|(\bar{A} - e^{-i\tau}\bar{B})(\bar{A} - e^{i\tau}\bar{B})|\psi\rangle \\ &= \Delta A^2 + \Delta B^2 - 2\mathrm{Re}(e^{i\tau}\langle\psi|\bar{A}\bar{B}|\psi\rangle). \quad (9)\end{aligned}$$

There exists $\tau = -\alpha$ so that $e^{i\tau}\langle\psi|\bar{A}\bar{B}|\psi\rangle$ is real, and it can be written as $|\langle\psi|\bar{A}\bar{B}|\psi\rangle|$. we obtain

$$\begin{aligned}&\langle\psi|(\bar{A} - e^{i\alpha}\bar{B})|\Pi|(\bar{A} - e^{-i\alpha}\bar{B})|\psi\rangle \\ &= \Delta A^2 + \Delta B^2 - 2|\langle\psi|\bar{A}\bar{B}|\psi\rangle| \\ &= \Delta A^2 + \Delta B^2 - |\langle[A,B]\rangle + \langle\{A,B\}\rangle - 2\langle A\rangle\langle B\rangle|. \quad (10)\end{aligned}$$

Since Π is the orthogonal complement to $|\psi\rangle\langle\psi|$, we can choose an arbitrary orthogonal decomposition of the projector Π ,

$$\Pi = \sum_{n=1}^{d-1} |\psi_n^\perp\rangle\langle\psi_n^\perp|, \quad (11)$$

where $\{|\psi>, |\psi_n^\perp>_{n=1}^{d-1}\}$ comprise an orthonormal complete basis in the d -dimensional Hilbert space. Whence, Eq. (10) can be rewritten as

$$\begin{aligned}&\sum_{n=1}^{d-1} |\langle\psi|(\bar{A} - e^{i\alpha}\bar{B})|\psi_n^\perp>|^2 \\ &= \sum_{n=1}^{d-1} |\langle\psi| A - e^{i\alpha}B |\psi_n^\perp\rangle|^2 \\ &= \Delta A^2 + \Delta B^2 - |\langle[A,B]\rangle + \langle\{A,B\}\rangle - 2\langle A\rangle\langle B\rangle|, \quad (12)\end{aligned}$$

which is equivalent to (8).

Uncertainty relation 2.

$$\Delta A^2\Delta B^2 = \frac{\left|\frac{1}{2}\langle[A,B]\rangle\right|^2 + \left|\frac{1}{2}\langle\{A,B\}\rangle - \langle A\rangle\langle B\rangle\right|^2}{\left(1 - \frac{1}{2}\sum_{n=1}^{d-1}\left|\langle\psi|\frac{A}{\Delta A} - e^{i\alpha}\frac{B}{\Delta B}|\psi_n^\perp\rangle\right|^2\right)^2}, \quad (13)$$

where $\{|\psi\rangle, |\psi_n^\perp\rangle_{n=1}^{d-1}\}$ comprise an orthonormal complete basis in the d -dimensional Hilbert space.

Proof To prove our uncertainty equality, let us define the operators $\Pi = I - |\psi\rangle\langle\psi|$, $\bar{A} = A - \langle A\rangle I$, and $\bar{B} = B - \langle B\rangle I$ and the unnormalized state $|\phi\rangle = \left(\frac{\bar{A}}{\Delta A} - e^{i\tau}\frac{\bar{B}}{\Delta B}\right)|\psi\rangle$. We have

$$\begin{aligned}&\langle\phi|\Pi|\phi\rangle \\ &= \langle\psi|\left(\frac{\bar{A}}{\Delta A} - e^{-i\tau}\frac{\bar{B}}{\Delta B}\right)|(I - |\psi\rangle\langle\psi|)|\left(\frac{\bar{A}}{\Delta A} - e^{i\tau}\frac{\bar{B}}{\Delta B}\right)|\psi\rangle \\ &= \langle\psi|\left(\frac{\bar{A}}{\Delta A} - e^{-i\tau}\frac{\bar{B}}{\Delta B}\right)\left(\frac{\bar{A}}{\Delta A} - e^{i\tau}\frac{\bar{B}}{\Delta B}\right)|\psi\rangle \\ &= 2 - 2\frac{\mathrm{Re}(e^{i\tau}\langle\psi|\bar{A}\bar{B}|\psi\rangle)}{\Delta A\Delta B}, \quad (14)\end{aligned}$$

There exists $\tau = -\alpha$ so that $e^{i\tau}\langle\psi|\bar{A}\bar{B}|\psi\rangle$ is real, and it can be written as $|\langle\psi|\bar{A}\bar{B}|\psi\rangle|$. We obtain

$$\begin{aligned}&\langle\psi|\left(\frac{\bar{A}}{\Delta A} - e^{i\alpha}\frac{\bar{B}}{\Delta B}\right)|\Pi|\left(\frac{\bar{A}}{\Delta A} - e^{-i\alpha}\frac{\bar{B}}{\Delta B}\right)|\psi\rangle \\ &= 2 - 2\frac{|\langle\psi|\bar{A}\bar{B}|\psi\rangle|}{\Delta A\Delta B}. \quad (15)\end{aligned}$$

Similarly, we choose the projector Π ,

$$\Pi = \sum_{n=1}^{d-1} |\psi_n^\perp\rangle\langle\psi_n^\perp|. \quad (16)$$

Then Eq. (15) can be rewritten as

$$\begin{aligned}&\sum_{n=1}^{d-1} \left|\langle\psi|\left(\frac{\bar{A}}{\Delta A} - e^{i\alpha}\frac{\bar{B}}{\Delta B}\right)|\psi_n^\perp\rangle\right|^2 \\ &= \sum_{n=1}^{d-1} \left|\langle\psi|\frac{A}{\Delta A} - e^{i\alpha}\frac{B}{\Delta B}|\psi_n^\perp\rangle\right|^2 \\ &= 2 - 2\frac{\left|\frac{1}{2}\langle[A,B]\rangle + \frac{1}{2}\langle\{A,B\}\rangle - \langle A\rangle\langle B\rangle\right|}{\Delta A\Delta B}, \quad (17)\end{aligned}$$

which is equivalent to (13).

The two uncertainty equalities (8) and (13) hold for all pairs of incompatible observables. If we retain only the term associated with $|\psi^\perp\rangle \in \{|\psi_n^\perp\rangle_{n=1}^{d-1}\}$ in the summation and discard the rest, the uncertainty equalities (8) and (13) reduce to the uncertainty relations (6) and (7), respectively.

2 Uncertainty relation in weak measurement

First proposed by Aharonov et al. [12], weak values are complex numbers so that one can define the weak value of A using two states: an initial state $|\psi\rangle$ called the pre-selection and a final state $|\varphi\rangle$ called the post-selection. The weak value of A has the form

$$\langle A\rangle_w = \frac{\langle\varphi|A|\psi\rangle}{\langle\varphi|\psi\rangle}. \tag{18}$$

For a given pre-selected and post-selected ensemble, we define the operator A_w as

$$A_w = \frac{\Pi_\varphi A}{p}, \tag{19}$$

where $\Pi_\varphi = |\varphi\rangle\langle\varphi|$ and $p = |\langle\varphi|\psi\rangle|^2$. The non-Hermitian operator has many properties [11] and is very useful in duality quantum computer [13-14].

Here, we construct an uncertainty relation in weak measurement for variances of two non-Hermitian operators A_w and B_w corresponding to two noncommuting observables A and B. The uncertainty relation quantitatively expresses the impossibility of jointly sharp preparation of pre- and post-selected (PPS) quantum states $|\psi\rangle$ and $|\varphi\rangle$ for the weak measurement of incompatible observables.

Uncertainty relation 3.

$$\Delta A_w^2 + \Delta B_w^2 \geqslant \left|\frac{1}{p}\langle\varphi|[A,B]|\varphi\rangle + \frac{1}{p}\langle\varphi|\{A,B\}|\varphi\rangle - 2\langle A_w\rangle\langle B_w\rangle^*\right| + \left|\langle\psi|A_w - e^{i\alpha}B_w|\psi^\perp\rangle\right|^2, \tag{20}$$

which is valid for two non-Hermitian operators A_w and B_w, where p is equivalent to $|\langle\varphi|\psi\rangle|^2$.

Proof To prove this relation we define the variance for any general (non-Hermitian) operator X in a state $|\psi\rangle$ which can be defined as in Refs. [15-16]

$$\Delta X^2 = \langle\psi|(X - \langle X\rangle)(X^\dagger - \langle X^\dagger\rangle)|\psi\rangle. \tag{21}$$

The variance of the non-Hermitian operation A_w in the quantum state $|\psi>$ can be defined as

$$\Delta A_w^2 = \langle\psi|(A_w - \langle A_w\rangle)(A_w^\dagger - \langle A_w^\dagger\rangle)|\psi\rangle, \tag{22}$$

where $\langle A_w\rangle = \langle\psi|A_w|\psi\rangle$ and $\langle A_w^\dagger\rangle = \langle\psi|A_w^\dagger|\psi\rangle = \langle A_w\rangle^*$. ΔA_w^2 can also be expressed as

$$\Delta A_w^2 = \langle\psi|A_w A_w^\dagger|\psi\rangle - \langle\psi|A_w|\psi\rangle\langle\psi|A_w^\dagger|\psi\rangle. \tag{23}$$

Similarly, for Hermitian operator B, we can define the operator

$$B_w = \frac{\Pi_\varphi B}{p}. \tag{24}$$

Then, the uncertainty for B_w can also be defined as

$$\Delta B_w^2 = \langle\psi|B_w B_w^\dagger|\psi\rangle - \langle\psi|B_w|\psi\rangle\langle\psi|B_w^\dagger|\psi\rangle. \tag{25}$$

To prove our uncertainty relation in weak measurement, we introduce a general inequality

$$\left\| C^\dagger|\psi\rangle - e^{i\tau}D^\dagger|\psi\rangle + k(|\psi\rangle - |\bar{\psi}\rangle)\right\|^2 \geqslant 0, \tag{26}$$

where $C^\dagger \equiv A_w^\dagger - \langle A_w^\dagger\rangle$ and $D^\dagger \equiv B_w^\dagger - \langle B_w^\dagger\rangle$. By expanding the square modulus, we have

$$\Delta A_w^2 + \Delta B_w^2 \geqslant -\lambda k^2 - \beta k + \pi, \tag{27}$$

where $\lambda \equiv 2(1 - \mathrm{Re}[\langle\psi|\bar{\psi}\rangle])$, $\pi \equiv 2\mathrm{Re}[e^{i\tau}\langle\psi|CD^\dagger|\psi\rangle]$, and $\beta \equiv 2\mathrm{Re}[\langle\psi|(-C + e^{-i\tau}D)|\bar{\psi}\rangle]$. By choosing the value of k that maximizes the right-hand-side of (27), namely $k = -\beta/2\lambda$, we get

$$\Delta A_w^2 + \Delta B_w^2 \geqslant \frac{\beta^2}{4\lambda} + \pi. \tag{28}$$

The above inequality can be rewritten as

$$\Delta A_w^2 + \Delta B_w^2 \geqslant \frac{\mathrm{Re}[\langle\psi|(-C + e^{-i\tau}D)|\bar{\psi}\rangle]^2}{2(1 - \mathrm{Re}[\langle\psi|\bar{\psi}\rangle])} + 2\mathrm{Re}[e^{i\tau}\langle\psi|CD^\dagger|\psi\rangle]. \tag{29}$$

Suppose $|\bar{\psi}\rangle = \cos\theta|\psi\rangle + e^{i\phi}\sin\theta|\psi^\perp\rangle$, where $|\psi^\perp\rangle$ is orthogonal to $|\psi\rangle$. By taking the limit $\theta \to 0$, the state $|\bar{\psi}\rangle$ reduces to $|\psi\rangle$ and then the above inequality can be reexpressed as

$$\Delta A_w^2 + \Delta B_w^2 \geqslant \mathrm{Re}[e^{i\phi}\langle\psi|(-A_w + e^{-i\tau}B_w)|\psi^\perp\rangle]^2 + 2\mathrm{Re}[e^{i\tau}\langle\psi|CD^\dagger|\psi\rangle]. \tag{30}$$

There exists $\tau = -\alpha$ so that $e^{i\tau}\langle\psi|CD^\dagger|\psi\rangle$ is real, and it can be written as $|\langle\psi|CD^\dagger|\psi\rangle|$. Then the second term becomes $\{\mathrm{Re}[e^{i\phi}\langle\psi| - A_w + e^{i\alpha}B_w|\psi^\perp\rangle]\}^2$. We can choose ϕ so that the term in square brackets is real and this term can be expressed as $|\langle\psi| A_w - e^{i\alpha} B_w |\psi^\perp\rangle|^2$. Whence, inequality (30) becomes

$$\Delta A_w^2 + \Delta B_w^2 \geqslant |\langle\psi| A_w - e^{i\alpha} B_w |\psi^\perp\rangle|^2 + 2|\langle\psi| C D^\dagger |\psi\rangle|. \quad (31)$$

The last term can be rewritten as

$$2|\langle C D^\dagger\rangle| = |\langle CD^\dagger + DC^\dagger + CD^\dagger - DC^\dagger\rangle|, \quad (32)$$

where

$$\langle CD^\dagger + DC^\dagger\rangle = \frac{1}{p}\langle\varphi| \{A,B\} |\varphi\rangle - \langle A_w\rangle\langle B_w\rangle^* - \langle A_w\rangle^*\langle B_w\rangle, \quad (33)$$

and

$$\langle CD^\dagger - DC^\dagger\rangle = \frac{1}{p}\langle\varphi| [A,B] |\varphi\rangle - \langle A_w\rangle\langle B_w\rangle^* + \langle A_w\rangle^*\langle B_w\rangle. \quad (34)$$

We combine Eqs. (33) and (34), Eq. (32) becomes

$$2|\langle CD^\dagger\rangle| = \left|\frac{1}{p}\langle\varphi| [A,B] |\varphi\rangle + \frac{1}{p}\langle\varphi| \{A,B\} |\varphi\rangle - 2\langle A_w\rangle\langle B_w\rangle^*\right|. \quad (35)$$

Combining Eqs. (32) and (35), we obtain the uncertainty relation (20).

3 Conclusions

In this work, we derived two new uncertainty equalities for the sum and product of variances of a pair of incompatible observables, which hold for all pairs of incompatible observables A and B. In fact, one can obtain a series of inequalities by retaining 1 to $(d-2)$ terms within the set $\{|\psi_n^\perp\rangle_{n=1}^{d-1}\}$. We also derived an uncertainty relation in weak measurement for two non-Hermitian operators A_w and B_w corresponding to two non-commuting observables A and B. The uncertainty relation quantitatively expresses the impossibility of jointly sharp preparation of PPS quantum states $|\psi\rangle$ and $|\varphi\rangle$ for measuring incompatible observables in weak measurement.

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