文章编号:2095-6134(2016)02-0145-05

Law of iterated logarithm of Galton-Watson processes in varying environment *

NAN Xiaojie†

(*School of Mathematical Sciences*, *University of Chinese Academy of Sciences*, *Beijing* 100049, *China*)

(Received 3 March 2015; Revised 18 May 2015)

Nan X J. Law of iterated logarithm of Galton-Watson processes in varying environment[J]. Journal of University of Chinese Academy of Sciences, 2016,33(2):145-149.

Abstract By the Berry-Esseen lemma and an important extension of the conditional Borel-Cantelli lemma (Asmussen, Trans Am Math Soc, 1977, 231: 233), we obtain the law of the iterated logarithm of the branching processes in varying environment under the condition that the second moment of the number of the offspring of each individual of each generation is uniformly upper/lower bounded. Further more, the condition is weaker than that of Gao(Gao, UCAS, Thesis 2011).

Key words varying environment; branching process; law of the iterated logarithm

CLC number:O211.4;O211.6 **Document code**: A **doi**:10.7523/j.issn.2095-6134.2016.02.001

变化环境中 Galton-Watson 过程的重对数律

南晓杰

(中国科学院大学数学科学学院,北京 100049)

摘 要 借助 Berry-Esseen 引理和 Asmussen 对条件 Borel-Cantelli 引理的重要推广,在变化环境中上临界分枝过程的每一代每一个个体的后代个体总数的 2 阶矩有一致上下界的情况下,得到变化环境中分枝过程的重对数律,从而改进了在相应的 $2+\delta$ 阶矩有限条件下的证明.

关键词 变化环境;分枝过程;重对数律

The law of the iterated logarithm(abbr. LIL) of the classical Galton-Watson process was firstly proved by Heyde[1] under the condition that the $2+\delta$ moment of the process is finite. In the same year, Heyde and Leslie[2] again obtained the LIL under the condition that the second moment is finite. Later, Asmussen[3] gave another proof via a very delicate truncation procedure and Kronecker lemma. The proof of Huggins[4] is based on the Skorohod embedding techniques and new properties of Brownian motion and stopping times.

Gao[5] proved the LIL of the super-critical Galton-Watson processes in varying environment satisfied that there is a uniform upper bound for the

* Supported by National Natural Science Foundation of China(11171342)

†E-mail:nanxiaojie12@mails.ucas.ac.cn

$2+\delta$ moment of the number of the offspring of each individual of each generation. In addition, the author pointed out a mistake in the proof of Theorem 1 in Heyde and Leslie[2]. Enlightened by the proof of the LIL of the classical Galton-Watson process in Asmussen[3], we obtain the LIL of the super-critical Galton-Watson processes in varying environment under the condition that the second moment has a uniform upper bound and a uniform lower bound.

1 Main result

Let $Z_0 \equiv 1$ and for all $n \geqslant 1$, define

$$Z_{n+1} = \begin{cases} \sum_{j=1}^{Z_n} X_{n,j}, & \text{if } Z_n \neq 0; \\ 0, & \text{if } Z_n = 0, \end{cases}$$

where $\{X_{n,j}; n \geqslant 0, j \geqslant 1\}$ are independent and for each $n \geqslant 0$, $\{X_{n,j}, j \geqslant 1\}$ have the same distribution $\{p_n(k), k \in \mathbb{N}\}$. $p_n(k)$ denotes the probability of k offspring produced by an individual of the n'th generation and $\mathbb{N}$ is the set of non-negative integers. Then $\{Z_n, n \geqslant 0\}$ is said to be a Galton-Watson process in varying environment (GWVE).

Let the generating functions of Z_n and $X_{n,j}$ are respectively $f_n(s)$ and $\phi_n(s)$, and let m_n and μ_n are respectively their expectations. Then

$$f_n(s) = \phi_0(\phi_1(\cdots\phi_{n-1}(s)\cdots)),$$

$$m_n = f'_n(1) = \prod_{i=0}^{n-1} \phi'_i(1) = \prod_{i=0}^{n-1} \mu_i.$$

From now on, we always assume that $\prod_{k=n}^{n-1} \mu_k = 1, 0 < m_n < \infty, \forall n \geqslant 0$. It is known that $\{W_n := Z_n / m_n, n \geqslant 0\}$ is a nonnegative martingale and there exists a nonnegative random variable W so that $\lim_{n\to\infty} W_n = W$ a.s.. Moreover, if $\sup_n \mathbb{E}(W_n^2) < \infty$, then $\mathbb{E}W = 1$ and $\sigma^2 := \mathrm{Var}(W) = \sum_{n=0}^{\infty} \delta_n^2/(\mu_n^2 m_n) < \infty$. These results can be found in Fearn[6].

Lemma 1.1 (Decomposition Lemma 1)

Let $\{Z_n, n \geqslant 0\}$ be a GWVE, then $\forall n \geqslant 0$, $r \geqslant 1$ we have

$$Z_{n+r} - m_{n,r} Z_n = \begin{cases} \sum_{j=1}^{Z_n} (Z_{n,r}^{(j)} - m_{n,r}), & \text{if } Z_n \neq 0; \\ 0, & \text{if } Z_n = 0, \end{cases}$$

where $Z_{n,r}^{(j)}$ represents the number of r'th generation offspring of the j'th of the Z_n individuals of the n'th generation, and $\{Z_{n,r}^{(j)}, j \geqslant 1\}$ are independent and identically distributed and independent of Z_n. Furthermore,

$$m_{n,r} := \mathbb{E}(Z_{n,r}^{(j)}) = \prod_{j=n}^{n+r-1} \mu_j,$$

$$\sigma_{n,r}^2 := \mathrm{Var}(Z_{n,r}^{(j)}) = (m_{n,r})^2 \sum_{j=n}^{n+r-1} \frac{\delta_j^2}{\mu_j^2 m_{n,j-n}}.$$

Proof See Ref. [5].

Remark 1.1 $Z_{n,1}^{(j)} = X_{n,j}, m_{0,r} = m_r, m_{n,1} = \mu_n, m_{n,0} = 1, \sigma_{0,r}^2 = \mathrm{Var}(Z_r)$ and $\sigma_{n,1}^2 = \delta_n^2$.

Lemma 1.2 (Decomposition Lemma 2)

Let $\{Z_n, n \geqslant 0\}$ be a GWVE, then $\forall n \geqslant 0$

$$m_n W - Z_n = \begin{cases} \sum_{j=1}^{Z_n} (W_n^{(j)} - 1), & \text{if } Z_n \neq 0; \\ 0, & \text{if } Z_n = 0, \end{cases}$$

where $\{W_n^{(j)}, j \geqslant 1\}$ are independent and identically distributed and independent of Z_n. If

$$\sum_{j=n}^{\infty} \frac{\delta_j^2}{\mu_j^2 \prod_{k=n}^{j-1} \mu_k} < \infty, \forall n \geqslant 0, \tag{1}$$

then

$$\mathbb{E}(W_n^{(j)}) = 1 \text{ and } \sigma_n^2 := \mathrm{Var}(W_n^{(j)}) = \sum_{j=n}^{\infty} \frac{\delta_j^2}{\mu_j^2 m_{n,j-n}}.$$

Proof See Ref. [5].

Remark 1.2 $W_0^{(1)} = W, \sigma_0^2 = \sigma^2 = \mathrm{Var}(W) = \sum_{j=0}^{\infty} \delta_j^2/(\mu_j^2 m_j)$.

Now assume that there exist four constants $\alpha, \beta, \tau, \gamma$ with $\beta > \alpha > 1$ and $\tau > \gamma > 0$ such that $\forall n \geqslant 0$

$$\alpha \leqslant \mu_n \leqslant \beta, \gamma^2 \leqslant \delta_n^2 \leqslant \tau^2, \tag{2}$$

$$\sum_{n=0}^{\infty} \int_{|y| > \zeta^n} y^2 \mathrm{d} F_n(y) < \infty, \tag{3}$$

$$\sum_{n=0}^{\infty} \int_{|y| > \zeta^n} y^2 \mathrm{d} G_n(y) < \infty, \tag{4}$$

where $1 < \zeta < \alpha^{1/4}$, and F_n is the distribution of $Z_{n,r}^{(j)} - m_{n,r}$ in Decomposition Lemma 1, and G_n is the distribution of $W_n^{(j)} - 1$ in Decomposition Lemma 2.

For any given $r \geqslant 1$, define

$Y_{n,j} := Z_{n,r}^{(j)} - m_{n,r}$,

$Y'_{n,j} := Y_{n,j} I(|Y_{n,j}| \leqslant \sqrt{m_n})$,

$$V_{n,j} := W_n^{(j)} - 1,$$

$$V'_{n,j} := V_{n,j} I(|V_{n,j}| \leqslant \sqrt{m_n}).$$

Theorem 1.1 Let $\{Z_n, n \geqslant 0\}$ be a GWVE. Suppose that $p_n(0) = 0, \forall n \geqslant 0$. If the conditions (2), (3), and (4) are satisfied, and

$$\mathrm{Var}(Y'_{n,j})/\mathrm{Var}(Y_{n,j}) \to 1, \text{ as } n \to \infty, \quad (5)$$

$$\mathrm{Var}(V'_{n,j})/\mathrm{Var}(V_{n,j}) \to 1, \text{ as } n \to \infty, \quad (6)$$

then for all $r \geqslant 1$, with probability one we have

$$\limsup_{n\to\infty}\left(\liminf_{n\to\infty}\right) \frac{Z_{n+r} - m_{n,r} Z_n}{(2\sigma_{n,r}^2 Z_n \log n)^{1/2}} = 1(-1), \quad (7)$$

$$\limsup_{n\to\infty}\left(\liminf_{n\to\infty}\right) \frac{m_n W - Z_n}{(2\sigma_n^2 Z_n \log n)^{1/2}} = 1(-1). \quad (8)$$

Remark 1.3 Since $p_n(0) > 0, \forall n \geqslant 0$, one has $W > 0$ a. s., hence $Z_n = O(m_n)$ a. s..

Remark 1.4 We can obtain $\alpha^n \leqslant m_n \leqslant \beta^n$ and Eq. (1) from the condition (2). According Remark 1.3, we know that $\log Z_n - n\log m \to \log W$ a. s., which means $\log\log Z_n/\log n \to 1$ a. s., so $\log n$ can be substituted by $\log\log Z_n$ in Eq. (7) and Eq. (8).

Remark 1.5 Since

$$\int_{|y| > \zeta^n} |y|^2 \mathrm{d} F_n(y) \leqslant \frac{1}{(\log \zeta^n)^{1+\delta}} \int_{|y| > \zeta^n} |y|^2 (\log |y|)^{1+\delta} \mathrm{d} F_n(y),$$

$$\int_{|y| > \zeta^n} |y|^2 \mathrm{d} G_n(y) \leqslant \frac{1}{(\log \zeta^n)^{1+\delta}} \int_{|y| > \zeta^n} |y|^2 (\log |y|)^{1+\delta} \mathrm{d} G_n(y),$$

where $0 < \delta < 1$, we can get the conditions (3) and (4) under the following conditions (9) and (10) are satisfied:

$$\sup_{n\geqslant 0} \int_{y\in\mathbb{R}} |y|^2 (\log |y|)^{1+\delta} \mathrm{d} F_n(y) < \infty, \quad (9)$$

$$\sup_{n\geqslant 0} \int_{y\in\mathbb{R}} |y|^2 (\log |y|)^{1+\delta} \mathrm{d} G_n(y) < \infty. \quad (10)$$

However (9) and (10) are weaker than (1.14) in Ref. [5].

Remark 1.6 The condition (5) holds naturally for a classical super-critical Galton-Watson branching process $\{Z_n, n \geqslant 0\}$ with $\mathbb{E}(Z_1 \log Z_1) < \infty$. Moreover, if there exists a random variable $Y \in L^2(\Omega, \mathscr{F}, \mathbb{P})$ so that $|Y_{n,1}| \leqslant Y$, then Eq. (5) can be deduced. Since $\sum_n \mathbb{P}(|Y_{n,1}| > \sqrt{m_n}) < \infty$, we almost surely have

$$Y'_{n,1} - Y_{n,1} \to 0 \text{ and } (Y'_{n,1})^2 - (Y_{n,1})^2 \to 0.$$

By the dominated convergence theorem, we have

$$\mathrm{Var}(Y'_{n,1}) - \mathrm{Var}(Y_{n,1}) = \mathrm{Var}(Y'_{n,j}) - \mathrm{Var}(Y_{n,j}) \to 0 \text{ a. s.},$$

hence the condition (5) holds. For Eq. (6) we have similar results.

2 Basic lemmas

Lemma 2.1 Let $\{\mathscr{F}_n, n \geqslant 0\}$ be an increasing sequence of σ-algebras and $\{T_n, n \geqslant 0\}$ a (not necessarily adapted) random variable sequence such that

$$\sum_{n=0}^{\infty} \triangle_n := \sum_{n=0}^{\infty} \sup_{y\in\mathbb{R}} |\mathbb{P}(T_n \leqslant y \mid \mathscr{F}_n) - \Phi(y)| < \infty,$$

where $\Phi(y)$ is the distribution function of $N(0,1)$. Then

$$\limsup_{n\to\infty} \frac{T_n}{(2\log n)^{1/2}} \leqslant 1 \text{ a. s.},$$

with the inequality replaced by equality if T_n is measurable with respect to $\mathscr{F}_{n+k}$ for some $1 \leqslant k < \infty$.

Proof See Ref. [3].

Lemma 2.2 (**Berry-Esseen Lemma**)
Let $\{X_n, n \geqslant 1\}$ be an independent and identically distributed random variable sequence such that $\mathbb{E}X_n = 0, \mathbb{E}X_n^2 = \sigma^2 > 0$ and $\mathbb{E}|X_n|^3 < \infty$. Denote $S_n := \sum_{k=1}^n X_k$. Then

$$\sup_{x\in\mathbb{R}} \left| \mathbb{P}\left(\frac{S_n}{\sigma\sqrt{n}} < x\right) - \Phi(x) \right| \leqslant A \frac{\mathbb{E}|X_1|^3}{\sigma^3 \sqrt{n}},$$

where $\Phi(x)$ is the standard normal distribution and A is a positive constant that is called the Berry-Esseen constant.

Proof See Ref. [7], P124.

Lemma 2.3 (**Kronecker Lemma**)
Let $\{b_n\}$ be an increasing sequence of positive real numbers with $b_n \to \infty$, and let $\{x_n\}$ be a sequence of real numbers with $\sum_{n=1}^{\infty} x_n = x$ (finite). Then

$$\frac{1}{b_n} \sum_{j=1}^{n} b_j x_j \to 0, \quad \text{as } n \to \infty.$$

Proof See Ref. [7], P63.

3 Proof of Theorem 1.1

Proof Denote $\mathscr{F}_0 := \sigma(Z_0)$ and $\mathscr{F}_n := \sigma\{X_{k,j};\ 0 \leqslant k \leqslant n-1, j \geqslant 1\}$. Then $\forall n \geqslant 1$, $\mathscr{F}_n$ is the σ-algebra generated by the individuals of previous $n-1$ generations. First prove Eq. (7). We only need to show

$$\limsup_{n\to\infty} \frac{Z_{n+r} - m_{n,r} Z_n}{(2\sigma_{n,r}^2 Z_n \log n)^{1/2}} = 1 \quad \text{a. s.} . \tag{11}$$

In fact, if Eq. (11) is true, let $\overline{Z_n} = -Z_n, n \geqslant 0$, then we have

$$\overline{Z_{n+r}} - m_{n,r}\overline{Z_n} = \sum_{j=1}^{Z_n} (-Y_{n,j}).$$

Repeating the proof of Eq. (11) for $\{-Y_{n,j}, n \geqslant 0, j \geqslant 1\}$ we obtain

$$\limsup_{n\to\infty} \frac{\overline{Z_{n+r}} - m_{n,r}\overline{Z_n}}{(2\sigma_{n,r}^2 Z_n \log n)^{1/2}} = 1 \quad \text{a. s.}, \tag{12}$$

which in fact is

$$\liminf_{n\to\infty} \frac{Z_{n+r} - m_{n,r} Z_n}{(2\sigma_{n,r}^2 Z_n \log n)^{1/2}} = -1 \quad \text{a. s.} .$$

Define

$$\widetilde{Y_{n,j}} := Y'_{n,j} - \mathbb{E}Y'_{n,j},$$

$$\widetilde{S_n} := \sum_{j=1}^{Z_n} \widetilde{Y_{n,j}},$$

$$\widetilde{\omega_n}^2 := \operatorname{Var}(\widetilde{S_n} \mid \mathscr{F}_n) = Z_n \operatorname{Var}(\widetilde{Y_{n,j}}),$$

$$T_n := \widetilde{S_n} / \widetilde{\omega_n}.$$

By a standard moment inequality,

$$\begin{aligned}
&\mathbb{E}(|\widetilde{Y_{n,j}}'|^3) \leqslant \\
&\mathbb{E}(|Y'_{n,j}|^3) + 3\,\mathbb{E}(|Y'_{n,j}|)(\mathbb{E}|Y'_{n,j}|)^2 + \\
&3\,\mathbb{E}(|Y'_{n,j}|)\,\mathbb{E}(Y'^2_{n,j}) + \mathbb{E}(|Y'_{n,j}|^3) \\
&\leqslant 8\,\mathbb{E}(|Y'_{n,j}|^3) = 8\int_{|y| \leqslant \sqrt{m_n}} |y|^3 \mathrm{d} F_n(y).
\end{aligned}$$

Letting A be the Berry-Essen constant, by the Berry-Esseen Lemma we have

$$\begin{aligned}
\triangle_n &:= \sup_{y\in\mathbb{R}} |\mathbb{P}(T_n \leqslant y \mid \mathscr{F}_n) - \Phi(y)| \\
&\leqslant 8A \frac{Z_n}{\widetilde{\omega_n}^3} \int_{|y| \leqslant \sqrt{m_n}} |y|^3 \mathrm{d} F_n(y).
\end{aligned} \tag{13}$$

By the condition (2) we can deduce that there exist a uniform upper and a uniform lower bound only dependent on r for $\sigma_{n,r}^2$. So there exist positive and finite constants C_1 and C_2 which are only dependent on r such that

$$C_1 \leqslant \liminf_{n\to\infty} \frac{Z_n}{\widetilde{\omega_n}^2} \leqslant \limsup_{n\to\infty} \frac{Z_n}{\widetilde{\omega_n}^2} \leqslant C_2. \tag{14}$$

From Remark 1.3 we know that $Z_n = O(m_n)$ a. s. , hence $\widetilde{\omega_n}^2 = O(m_n)$ a. s. . Combining with Eq. (13) and Eq. (14), we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{1}{\sqrt{m_n}} \int_{|y| \leqslant \sqrt{m_n}} |y|^3 \mathrm{d} F_n(y) \\
&= \sum_{n=0}^{\infty} \frac{1}{\sqrt{m_n}} \Big(\sum_{j=1}^{\zeta^n} \int_{j-1<|y|\leqslant j} |y|^3 \mathrm{d} F_n(y) \Big) + \\
&\sum_{n=0}^{\infty} \frac{1}{\sqrt{m_n}} \Big(\sum_{j=\zeta^n+1}^{\sqrt{m_n}} \int_{j-1<|y|\leqslant j} |y|^3 \mathrm{d} F_n(y) \Big) \\
&\leqslant \sum_{n=0}^{\infty} \sum_{j=1}^{\zeta^n} \frac{j}{\sqrt{m_n}} \int_{j-1<|y|\leqslant j} |y|^2 \mathrm{d} F_n(y) + \\
&\sum_{n=0}^{\infty} \sum_{j=\zeta^n+1}^{\sqrt{m_n}} \frac{j}{\sqrt{m_n}} \int_{j-1<|y|\leqslant j} |y|^2 \mathrm{d} F_n(y)) \\
&\leqslant \sum_{n=0}^{\infty} C_3 (\zeta^2/\sqrt{\alpha})^n + \sum_{n=0}^{\infty} \int_{|y|>\zeta^n} |y|^2 \mathrm{d} F_n(y),
\end{aligned} \tag{15}$$

where $C_3 > 0$ is a constant. By using the condition (3), Eq. (13) and Eq. (15) we have $\sum \triangle_n < \infty$ a. s. . Again applying Lemma 2.1 one eventually has

$$\limsup_{n\to\infty} \frac{T_n}{(2\log n)^{1/2}} \leqslant 1, \text{a. s.} .$$

In addition, since T_n is measurable with respect to $\mathscr{F}_{n+r}$, the above inequality should be replaced by equality.

$$S_n = \sum_{j=1}^{Z_n} Y_{n,j} = \sum_{j=1}^{Z_n} \{\widetilde{Y_{n,j}} + Y_{n,j} - Y'_{n,j} + \mathbb{E}Y'_{n,j}\}.$$

Thus it suffices to verify

$$\limsup_{n\to\infty} \frac{S_n}{(2\sigma_{n,r}^2 Z_n \log n)^{1/2}} = \limsup_{n\to\infty} \frac{T_n}{(2\log n)^{1/2}}.$$

Noting that

$$\frac{S_n}{(2\sigma_{n,r}^2 Z_n \log n)^{1/2}} = \frac{T_n}{(2\log n)^{1/2}} \left(\frac{\operatorname{Var}(Y'_{n,j})}{\sigma_{n,r}^2} \right)^{1/2} + \frac{\sum_{j=1}^{Z_n} \{Y_{n,j} - Y'_{n,j}\}}{(2\sigma_{n,r}^2 Z_n \log n)^{1/2}} + \frac{\sum_{j=1}^{Z_n} \mathbb{E}Y'_{n,j}}{(2\sigma_{n,r}^2 Z_n \log n)^{1/2}},$$

it suffices to verify that

$$\operatorname{Var}(Y'_{n,j}) / \sigma_{n,r}^2 \to 1, \tag{16}$$

$$\frac{\sum_{j=1}^{Z_n}\{Y_{n,j}-Y'_{n,j}\}}{(Z_n\log n)^{1/2}}\to 0, \quad (17)$$

$$\frac{\sum_{j=1}^{Z_n}\mathbb{E}Y'_{n,j}}{(Z_n\log n)^{1/2}}\to 0. \quad (18)$$

By the condition (5) we know that Eq. (16) holds. For Eq. (17) and Eq. (18), by Kronecer Lemma it only needs to prove that

$$\sum_{n=0}^{\infty}\frac{1}{\sqrt{m_n\log n}}\sum_{j=1}^{Z_n}|Y_{n,j}-Y'_{n,j}|<\infty, \quad (19)$$

$$\sum_{n=0}^{\infty}\frac{1}{\sqrt{m_n\log n}}\sum_{j=1}^{Z_n}|\mathbb{E}Y'_{n,j}|<\infty. \quad (20)$$

Since

$$\begin{aligned}|\mathbb{E}Y'_{n,j}| &= |\mathbb{E}(Y'_{n,j}-Y_{n,j})| \\ &\leqslant \mathbb{E}|Y'_{n,j}-Y_{n,j}| \\ &= \mathbb{E}|Y_{n,j}|I(|Y_{n,j}|>\sqrt{m_n}) \\ &= \int_{|y|>\sqrt{m_n}}|y|\,\mathrm{d}F_n(y),\end{aligned}$$

and noting that $Z_n=O(m_n)$ a. s. , it suffices for Eq. (19) and Eq. (20) that

$$\sum_{n=0}^{\infty}\frac{m_n}{\sqrt{m_n\log n}}\int_{|y|>\sqrt{m_n}}|y|\,\mathrm{d}F_n(y)<\infty \quad (21)$$

(for the first, taking the mean). And Eq. (21) certainly holds since even

$$\begin{aligned}&\sum_{n=0}^{\infty}\sqrt{m_n}\int_{|y|>\sqrt{m_n}}|y|\,\mathrm{d}F_n(y) \\ \leqslant &\sum_{n=0}^{\infty}\int_{|y|>\sqrt{m_n}}y^2\mathrm{d}F_n(y) \\ \leqslant &\sum_{n=0}^{\infty}\int_{|y|>\zeta^{2n}}|y|^2\mathrm{d}F_n(y)<\infty.\end{aligned}$$

Therefore, Eq. (17) and Eq. (18) hold.

In order to prove Eq. (8), recall $V_{n,j}:=W_n^{(j)}-1$. Repeating the proof of Eq. (11) and noting that there doesn't exist $1\leqslant r<\infty$ so that T_n is measurable with respect to $\mathscr{F}_{n+r}$, we have

$$\limsup_{n\to\infty}\frac{m_nW-Z_n}{(2\sigma_n^2Z_n\log n)^{1/2}}\leqslant 1 \text{ a. s. .} \quad (22)$$

Note that

$$\frac{m_n(W-W_n)}{(2\sigma_n^2Z_n\log n)^{1/2}}=\frac{m_{n+k}(W-W_{n+k})}{2\sigma_{n+k}^2Z_{n+k}\log(n+k)}\cdot\left(\frac{\sigma_{n+k}^2Z_{n+k}\log(n+k)}{\sigma_n^2Z_n\log n}\right)^{1/2}\frac{1}{m_{n,k}}+\frac{m_{n+k}(W_{n+k}-W_n)}{(2\sigma_{n,k}^2Z_n\log n)^{1/2}}\left(\frac{\sigma_{n,k}^2}{\sigma_n^2}\right)^{1/2}\frac{1}{m_{n,k}}.$$

So the lim sup part is at least

$$-\lim_{n\to\infty}\left(\frac{\sigma_{n+k}^2Z_{n+k}\log(n+k)}{\sigma_n^2Z_n\log n}\right)^{1/2}\frac{1}{m_{n,k}}+\lim_{n\to\infty}\left(\frac{\sigma_{n,k}^2}{\sigma_n^2}\right)^{1/2}\frac{1}{m_{n,k}}.$$

Again since the first item of the above expression is smaller than $C\alpha^{-k/2}$ and the second item

$$\frac{\sigma_{n,k}^2}{\sigma_n^2}\frac{1}{(m_{n,k})^2}=\left(\sum_{j=n}^{n+k-1}\frac{\delta_j^2}{\mu_j^2m_{n,j-n}}\right)\Big/\left(\sum_{j=n}^{\infty}\frac{\delta_j^2}{\mu_j^2m_{n,j-n}}\right)=\frac{1}{1+b_{n,k}},$$

where $b_{n,k}<C\alpha^{-k}$ and $C>0$ is a constant, the lim sup part of Eq. (8) is at least 1 as $k\to\infty$. The lim inf part of Eq. (8) can be proved in the same way as for Eq. (12). □

I am very grateful to my supervisor Prof. HU for intensive discussion with me and also benefit much from the communications with Gao Zhenlong, Liang Longyue, Zhang Zhiyang, and Zhao Rongjie.

References

[1] Heyde C C. Some almost sure convergence theorems for branching processes[J]. Z Wahrscheinlichkeitstheorie verw Geb, 1971, 20: 189-192.

[2] Heyde C C, Leslie J R. Improved classical limit analogues for Galton-Watson processes with or without immigration[J]. Bull Austral Math Soc, 1971, 5: 145-155.

[3] Asmussen S. Almost sure behavior of linear functionals of supercritical branching processes[J]. Transactions of The American Mathematical Society, 1977, 231(1): 233-248.

[4] Huggins R M. Laws of the iterated logarithm for time changed brownian motion with an application to branching processes [J]. Ann Probability, 1985, 13(4): 1148-1156.

[5] Gao Z L. Limit theorems for Galton-Watson processes in random environments[D]. Beijing: Graduate University of Chinese Academy of Sciences, 2011(in Chinese).

[6] Fearn D H. Galton-Watson processes with generation dependence [J]. Proc Sixth Berkeley Symp Math Statist Probab, 1971, 2: 159-172.

[7] Durrett R. Probability: theory and examples[M]. 3rd ed. Beijing: World Book Inc, 2011.