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Structure of $SK_1(\mathbb{Z}[C_4\times C_4], 2\mathbb{Z}[C_4\times C_4])$*

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Abstract In this paper we study the K-theory of the integral group ring $\mathbb{Z}[C_4\times C_4]$. We prove that the relative SK_1 group of the integral group ring $\mathbb{Z}[C_4\times C_4]$ is an elementary Abelian group of rank-3. We also show that the 4-rank of $K_2(\mathbb{Z}[C_4\times C_4])$ is at least 1 and the 2-rank of $K_2(\mathbb{Z}[C_4\times C_4])$ is at least 10.

Key words integral group ring; relative SK_1 group; K_2 group

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$SK_1(\mathbb{Z}[C_4\times C_4], 2\mathbb{Z}[C_4\times C_4])$的结构

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摘　要　主要研究整群环 $\mathbb{Z}[C_4\times C_4]$ 的 K 理论. 证明整群环 $\mathbb{Z}[C_4\times C_4]$ 的相对 SK_1 群为秩是 3 的初等阿贝尔群. 也证明了 $K_2(\mathbb{Z}[C_4\times C_4])$ 的 4 秩至少是 1, $K_2(\mathbb{Z}[C_4\times C_4])$ 的 2 秩至少是 10.

关键词　整群环; 相对 SK_1 群; K_2 群

The structure of the relative SK_1 group of the integral group ring is crucial to compute the K_2 group of the integral group ring. However it is very difficult to determine the exact structure of the relative SK_1 group. We can only find some results about the structure of the relative SK_1 group $SK_1(\mathbb{Z}[G])$ when G is an elementary Abelian p-group. In this paper, we consider the case of $G = C_4\times C_4 = <\sigma, \tau \mid \sigma^4 = \tau^4 = 1, \sigma\tau = \tau\sigma>$. We use the a result about the exponent of $SK_1(\mathbb{Z}[C_4\times C_4], 2\mathbb{Z}[C_4\times C_4])$ in Ref. [1] to determine the exact structure of the group $SK_1(\mathbb{Z}[C_4\times C_4], 2\mathbb{Z}[C_4\times C_4])$. We use the result about the structure of $K_2(\mathbb{F}_2[C_4\times C_4])$ in Ref. [2] to study the structure of $K_2(\mathbb{Z}[C_4\times C_4])$, and we obtain a lower bound of the 4-rank of $K_2(\mathbb{Z}[C_4\times C_4])$ and a lower bound of the 2-rank of $K_2(\mathbb{Z}[C_4\times C_4])$.

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1 Preliminaries

Let C_n be a cyclic group of order n, p be a prime number, and $\mathbb{F}_p$ be a finite field of p elements. Let J(R) denote the Jacobson radical of a ring R. We first introduce a basic definition which is crucial to compute the relative SK_1 group.

Definition 1.1 Let G be a finite Abelian p-group and O_F be the ring of integers in an algebraic number field F. A subset $S\subset \hat{G}$ is called an F-cluster of characters for G if S contains exactly one character for each simple $F[G]$-module. When $F=Q, O_F=\mathbb{Z}$, we will call S a cluster of characters for G. If S is a cluster of characters for G, We will define $S_0=\{\chi\in S \mid \mathbb{Z}[\chi]\neq\mathbb{Z}\}$. S_0 will be called an imaginarycluster of characters for G. Note that $S_0=S$-{trivial character} when p is odd and $S_0=\{\chi\in S \mid |\mathrm{im}(\chi)|>2\}$ when $p=2$.

The following two theorems will be used to determine $SK_1(\mathbb{Z}[C_4\times C_4], 2\mathbb{Z}[C_4\times C_4])$.

Theorem 1.1 (Theorem 1.10 in Ref. [1]) Let G be a finite Abelian 2-group and S_0 an imaginary cluster of characters for G. Then

$$SK_1(\mathbb{Z}[G],2\mathbb{Z}[G]) = [\prod_{\chi\in S_0}\mathrm{Im}\,\chi]/\psi_{\chi s_0}(G\otimes(2-\phi)(J(\hat{\mathbb{Z}}_2[G]))).$$

Note The elements $\sum\lambda_g g\in\hat{\mathbb{Z}}_2[G]$, ϕ is defined as follows:

$$\phi(\sum\lambda_g g)=\sum\lambda_g g^2.$$

$\forall g,h\in G$, $\psi_{\chi_{ij}}$ is defined as follows:

$$\psi_{\chi_{ij}}(g\otimes h)=\begin{cases}\chi_{ij}(g), & \text{if } \chi_{ij}(h)=1;\\ 1, & \text{if } \chi_{ij}(h)\neq 1.\end{cases}$$

Theorem 1.2 (Proposition 4.7 in Ref. [1]) $\exp(SK_1(\mathbb{Z}[C_4\times C_4],2\mathbb{Z}[C_4\times C_4]))=2$.

2 Main results

The following are the main results of this paper.

Theorem 2.1 $SK_1(\mathbb{Z}[C_4\times C_4],2\mathbb{Z}[C_4\times C_4])=C_2^3$.

Proof Let $G=C_4\times C_4=\langle\sigma,\tau\mid\sigma^4=\tau^4=1,\sigma\tau=\tau\sigma\rangle$, and ξ be a primitive 4th root of unity. Define the characters of $C_4\times C_4$ as follows:

$$\chi_{ij}, 0\leqslant i,j\leqslant 3,$$
$$\chi_{ij}(\sigma)=\xi^i, \chi_{ij}(\tau)=\xi^j,$$

then $\chi_{ij}(\sigma^h\tau^k)=\xi^{hi+jk}$. Let $S_0=\{\chi_{1j}, 0\leqslant j\leqslant 3, \chi_{i1}, i=0,2\}$, then by Proposition 4.7 of Ref. [1], S_0 is an imaginary cluster of $C_4\times C_4$.

Then by Theorem 1.1,

$$SK_1(\mathbb{Z}[C_4\times C_4],2\mathbb{Z}[C_4\times C_4]) = [\prod_{\chi\in S_0}\mathrm{im}\,\chi]/\psi_{\chi S_0}(G\otimes(2-\phi)(J(\hat{\mathbb{Z}}_2[G]))).$$

Obviously, $\mathrm{im}\,\chi_{01}=\mathrm{im}\,\chi_{10}=\mathrm{im}\,\chi_{11}=\mathrm{im}\,\chi_{12}=\mathrm{im}\,\chi_{13}=\mathrm{im}\,\chi_{21}=C_4$. In the following, we will fix the order of the product $\prod_{\chi\in S_0}\mathrm{im}\,\chi$ as $\prod_{\chi\in S_0}\mathrm{im}\,\chi=\mathrm{im}\,\chi_{01}\times\mathrm{im}\,\chi_{10}\times\mathrm{im}\,\chi_{11}\times\mathrm{im}\,\chi_{12}\times\mathrm{im}\,\chi_{13}\times\mathrm{im}\,\chi_{21}=C_4^6$.

Next we will determine the structure of $\psi_{\chi S_0}(G\otimes(2-\phi)(J(\hat{\mathbb{Z}}_2[G])))$.

For any $x\in J(\hat{\mathbb{Z}}_2[G])=\langle 2,g-1\mid g\in G\rangle$, we have $\psi_{\chi S_0}(\sigma\tau\otimes(2-\phi)(x))=\psi_{\chi S_0}(\sigma\otimes(2-\phi)(x))\cdot\psi_{\chi S_0}(\tau\otimes(2-\phi)(x))$, then $\psi_{\chi S_0}(G\otimes(2-\phi)(J(\hat{\mathbb{Z}}_2[G])))$ is generated by the following 34 elements,

$\{\psi_{\chi S_0}(\sigma\otimes(2-\phi)(2)), \psi_{\chi S_0}(\tau\otimes(2-\phi)(2)), \psi_{\chi S_0}(\sigma\otimes(2-\phi)(\sigma^i\tau^j-1)), \psi_{\chi S_0}(\tau\otimes(2-\phi)(\sigma^i\tau^j-1)), 0\leqslant i,j\leqslant 3\}$.

We will compute the value of $\psi_{\chi S_0}(\sigma\otimes(2-\phi)(\sigma\tau-1))$ as an example, and the same method can be used to determine the values of the other 33 elements. For

$$\begin{aligned}(2-\phi)(\sigma\tau-1)&=2\sigma\tau-2-\sigma^2\tau^2+1\\&=-\sigma^2\tau^2+2\sigma\tau-1,\end{aligned}$$

we have

$$\begin{aligned}&\psi_{\chi S_0}(\sigma\otimes(2-\phi)(\sigma\tau-1))\\&=\psi_{\chi S_0}(\sigma\otimes(-\sigma^2\tau^2+2\sigma\tau-1))\\&=(y_{ij})\in\prod_{\chi\in S_0}\mathrm{im}\,\chi,\end{aligned}$$

where

$$\begin{aligned}y_{ij}&=\psi_{\chi_{ij}}(\sigma\otimes(-\sigma^2\tau^2+2\sigma\tau-1))\\&=\psi_{\chi_{ij}}(\sigma\otimes\sigma^2\tau^2)^{-1}\psi_{\chi_{ij}}(\sigma\otimes\sigma\tau)^2\times\\&\quad\psi_{\chi_{ij}}(\sigma\otimes 1)^{-1}.\end{aligned}$$

By the definition of $\psi_{\chi_{ij}}$, we have $y_{01}=1, y_{10}=\xi^3, y_{11}=\xi^2, y_{12}=\xi^3, y_{13}=1$, and $y_{21}=\xi^2$.

Hence

$\psi_{\chi S_0}(\sigma \otimes (2-\phi)(\sigma\tau - 1)) = (1,\xi^3,\xi^2,\xi^3,1,\xi^2)$.

Using the same method, we can get the following generators:

$\psi_{\chi S_0}(\sigma \otimes (2-\phi)(1-1))$
$= \psi_{\chi S_0}(\tau \otimes (2-\phi)(1-1)) = (1,1,1,1,1,1)$,
$\{\psi_{\chi S_0}(\sigma \otimes (2-\phi)(2)) = (1,\xi^2,\xi^2,\xi^2,\xi^2,1)$,
$\psi_{\chi S_0}(\tau \otimes (2-\phi)(2)) = (\xi^2,1,\xi^2,1,\xi^2,\xi^2)$,
$\psi_{\chi S_0}(\sigma \otimes (2-\phi)(\tau-1)) = (1,1,\xi^3,\xi^2,\xi^3,\xi^2)$,
$\psi_{\chi S_0}(\sigma \otimes (2-\phi)(\tau^2-1)) = (1,1,\xi^2,1,\xi^2,1)$,
$\psi_{\chi S_0}(\sigma \otimes (2-\phi)(\tau^3-1)) = (1,1,\xi^3,\xi^2,\xi^3,\xi^2)$,
$\psi_{\chi S_0}(\sigma \otimes (2-\phi)(\sigma-1)) = (1,\xi^3,\xi^3,\xi^3,\xi^3,1)$,
$\psi_{\chi S_0}(\sigma \otimes (2-\phi)(\sigma\tau-1)) = (1,\xi^3,\xi^2,\xi^3,1,\xi^2)$,
$\psi_{\chi S_0}(\sigma \otimes (2-\phi)(\sigma\tau^2-1)) = (1,\xi^3,\xi^3,\xi^3,\xi^3,1)$,
$\psi_{\chi S_0}(\sigma \otimes (2-\phi)(\sigma\tau^3-1)) = (1,\xi^3,1,\xi^3,\xi^2,\xi^2)$,
$\psi_{\chi S_0}(\sigma \otimes (2-\phi)(\sigma^2-1)) = (1,\xi^2,\xi^2,\xi^2,\xi^2,1)$,
$\psi_{\chi S_0}(\sigma \otimes (2-\phi)(\sigma^2\tau-1)) = (1,\xi^2,\xi^3,1,\xi^3,\xi^2)$,
$\psi_{\chi S_0}(\sigma \otimes (2-\phi)(\sigma^2\tau^2-1)) = (1,\xi^2,1,\xi^2,1,1)$,
$\psi_{\chi S_0}(\sigma \otimes (2-\phi)(\sigma^2\tau^3-1)) = (1,\xi^2,\xi^3,1,\xi^3,\xi^2)$,
$\psi_{\chi S_0}(\sigma \otimes (2-\phi)(\sigma^3-1)) = (1,\xi^3,\xi^3,\xi^3,\xi^3,1)$,
$\psi_{\chi S_0}(\sigma \otimes (2-\phi)(\sigma^3\tau-1)) = (1,\xi^3\xi^3,1,\xi^3,\xi^2,\xi^2)$,
$\psi_{\chi S_0}(\sigma \otimes (2-\phi)(\sigma^3\tau^2-1)) = (1,\xi^3\xi^3,\xi^3,\xi^3,\xi^3,1)$,
$\psi_{\chi S_0}(\sigma \otimes (2-\phi)(\sigma^3\tau^3-1)) = (1,\xi^3,\xi^2,\xi^3,1,\xi^2)$,
$\psi_{\chi S_0}(\tau \otimes (2-\phi)(\tau-1)) = (\xi^3,1,\xi^3,1,\xi,\xi^3)$,
$\psi_{\chi S_0}(\tau \otimes (2-\phi)(\tau^2-1)) = (\xi^2,1,\xi^2,1,\xi^2,\xi^2)$,
$\psi_{\chi S_0}(\tau \otimes (2-\phi)(\tau^3-1)) = (\xi^3,1,\xi^3,1,\xi,\xi^3)$,
$\psi_{\chi S_0}(\tau \otimes (2-\phi)(\sigma-1)) = (1,1,\xi^3,\xi^2,\xi,\xi^2)$,
$\psi_{\chi S_0}(\tau \otimes (2-\phi)(\sigma\tau-1)) = (\xi^3,1,\xi^2,\xi^2,1,\xi^3)$,
$\psi_{\chi S_0}(\tau \otimes (2-\phi)(\sigma\tau^2-1)) = (\xi^2,1,\xi^3,\xi^2,\xi,1)$,
$\psi_{\chi S_0}(\tau \otimes (2-\phi)(\sigma\tau^3-1)) = (\xi^3,1,1,\xi^2,\xi^2,\xi^3)$,
$\psi_{\chi S_0}(\tau \otimes (2-\phi)(\sigma^2-1)) = (1,1,\xi^2,1,\xi^2,1)$,
$\psi_{\chi S_0}(\tau \otimes (2-\phi)(\sigma^2\tau-1)) = (\xi^3,1,\xi^3,1,\xi,\xi^3)$,
$\psi_{\chi S_0}(\tau \otimes (2-\phi)(\sigma^2\tau^2-1)) = (\xi^2,1,1,1,1,\xi^2)$,
$\psi_{\chi S_0}(\tau \otimes (2-\phi)(\sigma^2\tau^3-1)) = (\xi^3,1,\xi^3,1,\xi,\xi^3)$,
$\psi_{\chi S_0}(\tau \otimes (2-\phi)(\sigma^3-1)) = (1,1,\xi^3,\xi^2,\xi,\xi^2)$,
$\psi_{\chi S_0}(\tau \otimes (2-\phi)(\sigma^3\tau-1)) = (\xi^3,1,1,\xi^2,\xi^2,\xi^3)$,
$\psi_{\chi S_0}(\tau \otimes (2-\phi)(\sigma^3\tau^2-1)) = (\xi^2,1,\xi^3,\xi^2,\xi,1)$,
$\psi_{\chi S_0}(\tau \otimes (2-\phi)(\sigma^3\tau^3-1)) = (\xi^3,1,\xi^2,\xi^2,1,\xi^3)\}$.

Because some of these generators are the same, we use a_i to denote the different generators:

$\{a_1 = (1,\xi^2,\xi^2,\xi^2,\xi^2,1)$;
$a_2 = (\xi^2,1,\xi^2,1,\xi^2,\xi^2)$;
$a_3 = (1,1,\xi^3,\xi^2,\xi^3,\xi^2)$;
$a_4 = (1,1,\xi^2,1,\xi^2,1)$;
$a_5 = (1,\xi^3,\xi^3,\xi^3,\xi^3,1)$;
$a_6 = (1,\xi^3,\xi^2,\xi^3,1,\xi^2)$;
$a_7 = (1,\xi^3,1,\xi^3,\xi^2,\xi^2)$;
$a_8 = (1,\xi^2,\xi^3,1,\xi^3,\xi^2)$;
$a_9 = (1,\xi^2,1,\xi^2,1,1)$;
$a_{10} = (\xi^3,1,\xi^3,1,\xi,\xi^3)$;
$a_{11} = (1,1,\xi^3,\xi^2,\xi,\xi^2)$;
$a_{12} = (\xi^3,1,\xi^2,\xi^2,1,\xi^3)$;
$a_{13} = (\xi^2,1,\xi^3,\xi^2,\xi,1)$;
$a_{14} = (\xi^3,1,1,\xi^2,\xi^2,\xi^3)$;
$a_{15} = (\xi^2,1,1,1,1,\xi^2)$;
$a_{16} = (\xi^3,1,\xi^2,\xi^2,1,\xi^3).\}$

Next we will determine the structure of the group generated by a_i which is $\psi_{\chi S_0}(G \otimes (2-\phi)(J(\hat{\mathbb{Z}}_2[G])))$.

For $1 \leqslant i \leqslant 6$, let b_i be the vector of dimension 6 in which the ith component is ξ^2 and the other components are all equal to 1. Let $b_7 = (\xi,1,1,1,1,\xi)$, $b_8 = (1,\xi,1,\xi,1,1)$, $b_9 = (1,1,\xi,1,\xi,1)$. We show that $\{b_i, 1 \leqslant i \leqslant 9\}$ is a generating set of the group generated by $\{a_i, 1 \leqslant i \leqslant 16\}$.

On one hand, $\{b_i, 1 \leqslant i \leqslant 9\}$ can be generated by $\{a_i, 1 \leqslant i \leqslant 16\}$:

$$a_5 \cdot a_6 \cdot a_{11} = (1,\xi^2,1,1,1,1) = b_2,$$
$$a_9 \cdot b_2 = (1,1,1,\xi^2,1,1) = b_4,$$
$$a_5 \cdot a_7 \cdot a_8 \cdot b_4 = (1,1,\xi^2,1,1,1) = b_3,$$
$$a_4 \cdot b_3 = (1,1,1,1,\xi^2,1) = b_5,$$
$$a_3 \cdot a_4 \cdot a_{10} \cdot a_{12} \cdot a_{14} \cdot a_{16} = (1,1,1,1,1,\xi^2) = b_6,$$
$$a_{15} \cdot b_6 = (\xi^2,1,1,1,1,1) = b_1,$$
$$a_{12} \cdot b_1 \cdot b_3 \cdot b_4 \cdot b_6 = (\xi,1,1,1,1,\xi) = b_7,$$
$$a_7 \cdot b_2 \cdot b_4 \cdot b_5 \cdot b_6 = (1,\xi,1,\xi,1,1) = b_8,$$
$$a_8 \cdot b_2 \cdot b_3 \cdot b_5 \cdot b_6 = (1,1,\xi,1,\xi,1) = b_9.$$

On the other hand, $\{a_i, 1 \leqslant i \leqslant 16\}$ can be generated by $\{b_i, 1 \leqslant i \leqslant 9\}$:

$$a_1 = b_2 \cdot b_3 \cdot b_4 \cdot b_5, a_2 = b_1 \cdot b_3 \cdot b_5 \cdot b_6,$$
$$a_3 = b_3 \cdot b_4 \cdot b_5 \cdot b_6 \cdot b_9, a_4 = b_3 \cdot b_5,$$
$$a_5 = b_2 \cdot b_3 \cdot b_4 \cdot b_5 \cdot b_8 \cdot b_9, a_6$$

$$= b_2 \cdot b_3 \cdot b_4 \cdot b_6 \cdot b_8,$$

$$a_7 = b_2 \cdot b_4 \cdot b_5 \cdot b_6 \cdot b_8, a_8 = b_2 \cdot b_3 \cdot b_5 \cdot b_6 \cdot b_9,$$

$$a_9 = b_2 \cdot b_4, a_{10} = b_1 \cdot b_3 \cdot b_6 \cdot b_7 \cdot b_9,$$

$$a_{11} = b_3 \cdot b_4 \cdot b_6 \cdot b_9, a_{12} = b_1 \cdot b_3 \cdot b_4 \cdot b_6 \cdot b_7,$$

$$a_{13} = b_1 \cdot b_3 \cdot b_4 \cdot b_9, a_{14} = b_1 \cdot b_4 \cdot b_5 \cdot b_6 \cdot b_7,$$

$$a_{15} = b_1 \cdot b_6, a_{16} = b_1 \cdot b_3 \cdot b_4 \cdot b_6 \cdot b_7.$$

Hence $\{b_i, 1 \leqslant i \leqslant 9\}$ is an generating set of $\psi_{\chi S_0}(G \otimes (2-\phi)(J(\hat{\mathbb{Z}}_2[G])))$.

It is easy to know that these 6 elements $\{b_i, 1 \leqslant i \leqslant 6\}$ generate an elementary Abelian 2-group of rank 6. We denote this group by H. Then the 8 cosets,

$\{H, b_7H, b_8H, b_9H, b_7b_8H, b_7b_9H, b_8b_9H, b_7b_8b_9H\}$,

are disjoint with each other and their union is $\psi_{\chi S_0}(G \otimes (2-\phi)(J(\hat{\mathbb{Z}}_2[G])))$. Hence

$$|\psi_{\chi S_0}(G \otimes (2-\phi)(J(\hat{\mathbb{Z}}_2[G])))| = 2^6 \cdot 8 = 2^9,$$

and

$$|SK_1(\mathbb{Z}[C_4 \times C_4], 2\mathbb{Z}[C_4 \times C_4])| = \frac{4^6}{2^9} = 2^3.$$

By Theorem 1.2,

$$\exp(SK_1(\mathbb{Z}[C_4 \times C_4], 2\mathbb{Z}[C_4 \times C_4]) = 2.$$

So

$$SK_1(\mathbb{Z}[C_4 \times C_4], 2\mathbb{Z}[C_4 \times C_4]) = C_2^3.$$

□

Corollary 2.1 $SK_1(\mathbb{Z}[C_4 \times C_4 \times C_2]) = C_2^4$.

Proof By Theorem 1.11 in Ref. [1], we have $SK_1(\mathbb{Z}[C_4 \times C_4 \times C_2]) \simeq SK_1(\mathbb{Z}[C_4 \times C_4) \oplus SK_1(\mathbb{Z}[C_4 \times C_4], 2\mathbb{Z}[C_4 \times C_4])$. By Theorem 5.5 in Ref. [1], $SK_1(\mathbb{Z}[C_4 \times C_4]) = C_2$. Then by Theorem 2.1, we have

$$SK_1(\mathbb{Z}[C_4 \times C_4], 2\mathbb{Z}[C_4 \times C_4]) = C_2^3.$$

So $SK_1(\mathbb{Z}[C_4 \times C_4 \times C_2]) = C_2^4$. □

Theorem 2.2 The 4-rank of $K_2(\mathbb{Z}[C_4 \times C_4])$ is at least 1 and its 2-rank is at least 10.

Proof By the long exact sequence of K-theory, we have

$$K_2(\mathbb{Z}[C_4 \times C_4]) \xrightarrow{f_1} K_2(\mathbb{F}_2[C_4 \times C_4]) \xrightarrow{f_2} SK_1(\mathbb{Z}[C_4 \times C_4], 2\mathbb{Z}[C_4 \times C_4]) \xrightarrow{f_3} SK_1(\mathbb{Z}[C_4 \times C_4]) \xrightarrow{f_4} SK_1(F_2[C_4 \times C_4]).$$

By Theorem 1.2 in Ref. [2], $K_2(F_2[C_4 \times C_4]) = C_2^9 \oplus C_4^3$. By Theorem 2.1, $SK_1(\mathbb{Z}[C_4 \times C_4], 2\mathbb{Z}[C_4 \times C_4]) = C_2^3$. By Theorem 5.5 in Ref. [1], we have $SK_1(\mathbb{Z}[C_4 \times C_4]) = C_2$. Then the exact sequence becomes

$$K_2(\mathbb{Z}[C_4 \times C_4]) \xrightarrow{f_1} C_2^9 \oplus C_4^3 \xrightarrow{f_2} C_2^3 \xrightarrow{f_3} C_2 \xrightarrow{f_4} 1.$$

By the exactness, $\mathrm{im}(f_2) = \ker(f_3) = C_2^2$. Then we get the following exact sequence,

$$K_2(\mathbb{Z}[C_4 \times C_4]) \xrightarrow{f_1} C_2^9 \oplus C_4^3 \xrightarrow{f_2} C_2^2 \longrightarrow 1.$$

By the exactness, $\mathrm{im}(f_1) = \ker(f_2)$ can only be one of the following three cases,

$$\{C_2^{11} \oplus C_4; C_2^9 \oplus C_4^2; C_2^7 \oplus C_4^3\}.$$

In any of these cases, $\mathrm{im}(f_1)$ contains one cyclic subgroup of order 4 as its direct summand. Then the 4-rank of $K_2(\mathbb{Z}[C_4 \times C_4])$ is at least 1 and the 2-rank is at least 10. □

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