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Full implicit schemes for stochastic differential equations with one noise*

SUN Liying, WANG Lijin†

(*School of Mathematical Sciences*, *University of Chinese Academy of Sciences*, *Beijing* 100049, *China*)

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Abstract Two kinds of full implicit numerical schemes for stochastic differential equations in the Itô sense are given via construction of the numerical methods for the equivalent stochastic differential equations in the sense of Stratonovich and Backward-Itô. This approach could be applied to the stochastic differential equations with one noise, and we prove that the two methods which are generated by the equivalent stochastic differential equations are of mean-square order 1.

Key words stochastic differential equations; full implicit numerical schemes; stochastic differential equations in the Itô sense; stochastic differential equations in the Stratonovich sense

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带有一个噪声的随机微分方程的全隐式格式

孙丽莹, 王丽瑾

(中国科学院大学数学科学学院, 北京 100049)

摘 要 通过将 Itô 型随机微分方程转换为等价的 Stratonovich 型和向后 Itô 型随机微分方程,构造出分别与 Stratonovich 型和向后 Itô 型随机微分方程相容的两种全隐式随机数值格式.这两种数值格式应用于带一个噪声的随机微分方程,且均为均方 1 阶收敛的.

关键词 随机微分方程; 全隐式数值格式; Itô 型随机微分方程; Stratonovich 型随机微分方程

Consider the numerical approximations for stochastic differential equations (SDEs) in the Itô sense as follows

$$\begin{cases} \mathrm{d}X = a(t,X)\,\mathrm{d}t + b(t,X)\,\mathrm{d}W(t), \\ X(t_0) = X_0, \end{cases} \tag{1}$$

where X, $a(t, x^1, \cdots, x^n)$, $b(t, x^1, \cdots, x^n)$ are n-dimensional column-vectors and $W(t)$ is a standard Wiener process. In addition, we suppose that the coefficients $a(t, x^1, \cdots, x^n)$ and $b(t, x^1, \cdots, x^n)$ are sufficiently smooth functions that guarantee the existence and the uniqueness of the solution in the interval $[t_0, t_0 + T]$ (see Refs. [1-2] for details).

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†Corresponding author, E-mail: ljwang@ucas.ac.cn

Let $X(t;t_0,x,\omega)$, $t_0 \leqslant t \leqslant t_0+T$, be the solution of the SDE (1) where ω is an elementary event. Moreover, we denote by X_k, $k=0,\cdots,N$, $t_{k+1}-t_k=h$, $t_N=t_0+T$, the numerical method for (1) based on the one-step approximation $\overline{X}=\overline{X}(t+h;t,x)$.

As is known in Refs. [3-6], there are many kinds of numerical schemes for the SDE (1). However, the construction of these approximations are usually based on the SDEs in the Itô sense. Hence, it is meaningful to investigate whether we could construct the methods by using their equivalent SDEs in the sense of Stratonovich and Backward-Itô.

1 Full implicit schemes

1.1 Equivalent equations in the sense of Stratonovich and Backward-Itô

As is known in Refs. [1-2, 5-6], the equivalent equation with respect to the SDE (1) in the sense of Stratonovich is as follows:

$$\mathrm{d}X=\left(\left(a-\frac{1}{2}\frac{\partial b}{\partial x}b\right)(t,X)\right)\mathrm{d}t+b(t,X)\circ\mathrm{d}W(t), \tag{2}$$

where $X(t_0)=X_0$ and $\frac{\partial b}{\partial x}$ is the matrix with entry $\frac{\partial b^i}{\partial x^j}$ at the intersection of the ith row and jth column. In addition, we also require that the column vectors $\frac{\partial b}{\partial x}b$ satisfy a uniform Lipschitz condition with respect to $x\in R^n$.

Based on the relationship between the Itô integral and the Stratonovich integral, we can obtain the SDE (2). Analogously, we can define another stochastic integral, Backward-Itô integral, and get the equivalent equation of SDE (1) in the sense of Backward-Itô. Let $(\Omega,\mathscr{F})$ be a measure space with the probability P and $\mathscr{F}_t$ be the σ-algebra generated by the n-dimensional Brownian motions $B_i(s)_{(1\leqslant i\leqslant n,0\leqslant s\leqslant t)}$. We denote by $V(S,T)$ the class of functions

$$f(t,\omega):[0,\infty)\times\Omega\to\mathbb{R},$$

such that

1) $(t,\omega)\to f(t,\omega)$ is $\mathbb{B}\times\mathscr{F}$-measurable, where $\mathbb{B}$ denotes the Borel σ-algebra on $[0,\infty)$;

2) $f(t,\omega)$ is $\mathscr{F}_t$-adapted and $E\left[\int_s^t f(\theta,\omega)^2\mathrm{d}\theta\right]\leqslant\infty$.

Furthermore, let $L^2(P)$ be a Hilbert space with the following inner product

$$(X,Y)_{L^2(P)}:=E[X\cdot Y];X,Y\in L^2(P).$$

Definition 1.1 Suppose that $f\in V(0,T)$ and $t\to f(t,\omega)$ is continuous for a.e. ω. Then the Backward-Itô integral of f is defined by

$$\int_0^T f(t,\omega)*\mathrm{d}W(t)=\lim_{h\to 0}\sum_j f(t_{j+1},\omega)\Delta W_j,$$

whenever the limit exists in $L^2(P)$.

Note that this kind of integral has been introduced in Ref. [3]. For convenience, we give it the name Backward-Itô integral. According to the definition of the Backward-Itô integral, we can easily get the relationship between the Itô integral and the Backward-Itô integral as follows:

$$\int_0^T f(t)*\mathrm{d}W(t)=\int_0^T f(t)\,\mathrm{d}W(t)+\int_0^T\frac{\partial f(t)}{\partial W}\mathrm{d}t, \tag{3}$$

where $f\in V(0,T)$ and $t\to f(t,\omega)$ is continuous for a.e. ω. Then using formula (3), we can obtain the equivalent SDE in the sense of Backward-Itô

$$\mathrm{d}X=\left(\left(a-\frac{\partial b}{\partial x}b\right)(t,X)\right)\mathrm{d}t+b(t,X)*\mathrm{d}W(t), \tag{4}$$

where $X(t_0)=X_0$.

1.2 The convergence theorem on mean-square methods from Ref. [4]

Theorem 1.1 See Ref. [4]. Suppose that the one-step approximation $\overline{X}(t+h;t,x)$ has the order of accuracy p_1 for the mathematical expectation of the deviation and order of accuracy p_2 for the mean-square deviation. More precisely, for arbitrary $t_0\leqslant t\leqslant t_0+T-h$, $x\in R^n$ the following inequalities hold:

$$|\boldsymbol{E}((X-\overline{X})(t+h;t,x))|\leqslant K(1+|x|^2)^{\frac{1}{2}}h^{p_1},$$

$$[\boldsymbol{E}|(X-\overline{X})(t+h;t,x)|^2]^{\frac{1}{2}}\leqslant K(1+|x|^2)^{\frac{1}{2}}h^{p_2}, \tag{5}$$

Also, let

$$p_2 \geqslant \frac{1}{2}, p_1 \geqslant p_2 + \frac{1}{2}.$$

Then for any N and $k = 0, \cdots, N$ the following inequality holds:

$$[\boldsymbol{E} \mid (X - \bar{X})(t_k; t_0, X_0) \mid^2]^{\frac{1}{2}} \leqslant K(1 + \mid X_0 \mid^2)^{\frac{1}{2}} h^{p_2 - \frac{1}{2}}, \tag{6}$$

i. e. the order of accuracy of the method constructed using the one-step approximation $\bar{X}(t + h; t, x)$ is $p = p_2 - \frac{1}{2}$.

Theorem 1.2 See Refs. [5-6]. Let the one-step approximation $\bar{X}(t + h; t, x)$ satisfy the condition of Theorem 2.1. Suppose that $\tilde{X}(t + h; t, x)$ is such that

$$\mid \boldsymbol{E}(\bar{X}(t + h; t, x) - \tilde{X}(t + h; t, x)) = O(h^{p_1}),$$

$$[\boldsymbol{E} \mid \bar{X}(t + h; t, x) - \tilde{X}(t + h; t, x) \mid^2]^{\frac{1}{2}} = O(h^{p_2}), \tag{7}$$

with the same h^{p_1} and h^{p_2}. Then the method based on the one-step approximation $\tilde{X}(t + h; t, x)$ has the same mean-square order of accuracy as the method based on $\bar{X}(t + h; t, x)$, i. e., its order is equal to $p = p_2 - \frac{1}{2}$.

Besides, the authors of Refs. [4-6] have mentioned that the increments of Wiener processes should be substituted by truncated random variables in implicit schemes. In detail, $\Delta w(h) = \xi \sqrt{h}$, where ξ is $N(0,1)$-distributed random variable, is replaced by another random variable $\zeta_h \sqrt{h}$ such that $\zeta_h \sqrt{h}$ is bounded, and

$$\zeta_h = \begin{cases} \xi, & \text{if} \quad \mid \xi \mid \leqslant A_h, \\ A_h, & \text{if} \quad \xi > A_h, \\ -A_h, & \text{if} \quad \xi < A_h, \end{cases} \tag{8}$$

where $A_h = \sqrt{2l \mid \ln h \mid}$, $l \geqslant 1$.

1.3 Full implicit schemes

First, we consider the SDE (2). As is known, the exact solution is

$$X(t + h; t, x) = x + \int_t^{t+h} b(s, X(s)) \circ \mathrm{d}W(s) + \int_t^{t+h} \left[\left(a - \frac{1}{2} \frac{\partial b}{\partial x} b \right)(s, X(s)) \right] \mathrm{d}s, \tag{9}$$

which coincides with the solution of the SDE (1) by using the Stratonovich integrals. Let the mid-rectangle formula approximate the integrals in (9). We can get a full implicit scheme of the first mean-square order as follows

$$X_{k+1} = X_k + b\left(t_k + \frac{h}{2}, \frac{X_k + X_{k+1}}{2}\right)\zeta_h \sqrt{h} + \left[\left(a - \frac{1}{2} \frac{\partial b}{\partial x} b \right)\left(t_k + \frac{h}{2}, \frac{X_k + X_{k+1}}{2} \right) \right] h. \tag{10}$$

Similarly, the solution of the SDE (4) is

$$X(t + h; t, x) = x + \int_t^{t+h} b(s, X(s)) * \mathrm{d}W(s) + \int_t^{t+h} \left[\left(a - \frac{\partial b}{\partial x} b \right)(s, X(s)) \right] \mathrm{d}s, \tag{11}$$

which is also in agreement with the solution of the SDE (1). Similarly, if the integrals in (11) are approximated by the right-rectangle formula, the numerical scheme has the form:

$$X_{k+1} = X_k + b(t_k + h, X_{k+1}) \zeta_h \sqrt{h} + [(a - \frac{\partial b}{\partial x} b)(t_k + h, X_{k+1})] h,$$

which we can prove to be of mean-square order 0.5. However, based on an analog of Taylor expansion of the solution (11), another full implicit method

$$X_{k+1} = X_k + \left[\left(a - \frac{1}{2} \frac{\partial b}{\partial x} b \right)(t_{k+1}, X_{k+1}) \right] h + \left[\frac{1}{2} \frac{\partial b}{\partial x} b(t_{k+1}, X_{k+1}) \right] \zeta_h^2 h + b(t_k, X_k) \zeta_h \sqrt{h}, \tag{12}$$

which is of the first mean-square order, could be constructed.

2 Mean-square order of the full implicit schemes

Theorem 2.1 The numerical method (10) with $A_h = \sqrt{4 \mid \ln h \mid}$ is of mean-square order 1.

Proof According to an analog of Taylor expansion of the solution (9), the one-step approximation $\bar{X}(t + h; t, x)$ as follows

$$\bar{X} = x + \left[\left(a - \frac{1}{2} \frac{\partial b}{\partial x} b \right)\left(t + \frac{h}{2}, x \right) \right] h +$$

$$b(t,x)\Delta W(h) + \frac{1}{2}\frac{\partial b}{\partial x}(t,x)b(t,x)(\Delta W(h))^2. \tag{13}$$

has the first mean-square order of convergence under appropriate conditions. The full implicit method (10) we propose is

$$\widetilde{X} = x + \left[\left(a - \frac{1}{2}\frac{\partial b}{\partial x}b\right)\left(t + \frac{h}{2}, \frac{x + \widetilde{X}}{2}\right)\right]h + b\left(t + \frac{h}{2}, \frac{x + \widetilde{X}}{2}\right)\zeta_h\sqrt{h}. \tag{14}$$

After expanding the right side of (14) about x, we get

$$\widetilde{X} = x + \left[\left(a - \frac{1}{2}\frac{\partial b}{\partial x}b\right)\left(t + \frac{h}{2}, x\right)\right]h + b(t,x)\zeta_h\sqrt{h} + \frac{1}{2}\frac{\partial b}{\partial x}(t,x)b(t,x)\zeta_h^2 h + \rho_1. \tag{15}$$

It is obvious that $|\boldsymbol{E}(\rho_1)| = O(h^2)$ and $\boldsymbol{E}(\rho_1)^2 = O(h^3)$. Since

$$\overline{X} - \widetilde{X} = b(t,x)(\Delta W(h) - \zeta_h\sqrt{h}) + \frac{1}{2}\frac{\partial b}{\partial x}(t,x)b(t,x)(\xi^2 - \zeta_h^2)h - \rho_1, \tag{16}$$

it is possible to show that $|\boldsymbol{E}(\overline{X} - \widetilde{X})| = O(h^2)$ and

$$E(\overline{X} - \widetilde{X})^2 \leqslant K\boldsymbol{E}(\xi - \zeta_h)^2 h + Kh^2\boldsymbol{E}(\xi^2 - \zeta_h^2)^2 + O(h^3), \tag{17}$$

where K is a sufficiently large constant. Using the inequalities $\boldsymbol{E}(\xi - \zeta_h)^2 = O(h^2)$ and $\boldsymbol{E}(\xi^2 - \zeta_h^2)^2 \leqslant 27h$ in Ref. [7], we can easily prove $\boldsymbol{E}(\overline{X} - \widetilde{X})^2 \leqslant O(h^3)$. Finally, by applying Theorem 2.2, we prove the theorem. □

Theorem 2.2 The numerical method (12) with $A_h = \sqrt{4|\ln h|}$ is of mean-square order 1.

Proof Considering the numerical method (12), we expand the right side of the solution (11) and obtain

$$X(t+h;t,x) = x + \int_t^{t+h} b(s,X(s)) * \mathrm{d}W(s) + \int_t^{t+h}\left[\left(a - \frac{\partial b}{\partial x}b\right)(s,X(s))\right]\mathrm{d}s$$

$$= x + \left[\left(a - \frac{\partial b}{\partial x}b\right)(t,x)\right]h + b(t,x)\Delta W(h) + \int_t^{t+h}\int_t^s Lb(\theta,X(\theta))\mathrm{d}\theta * \mathrm{d}W(s) + \int_t^{t+h}\int_t^s \frac{\partial b}{\partial x}b(\theta,X(\theta))\mathrm{d}W(\theta) * \mathrm{d}W(s) + \rho^1$$

$$= x + \left[\left(a - \frac{\partial b}{\partial x}b\right)(t,x)\right]h + b(t,x)\Delta W(h) + \int_t^{t+h}\int_t^s \frac{\partial b}{\partial x}b(t,x)\mathrm{d}W(\theta) * \mathrm{d}W(s) + \rho^2, \tag{18}$$

where $|\boldsymbol{E}(\rho^1)| = O(h^2)$, $\boldsymbol{E}(\rho^1)^2 = O(h^3)$, $|\boldsymbol{E}(\rho^2)| = O(h^2)$, and $\boldsymbol{E}(\rho^2)^2 = O(h^3)$. Since $\int_t^{t+h}\int_t^s \mathrm{d}W(\theta) * \mathrm{d}W(s) = \frac{1}{2}\Delta W(h)^2 + \frac{h}{2}$, we can get

$$X(t+h;t,x) = x + \left[\left(a - \frac{\partial b}{\partial x}b\right)(t,x)\right]h + \frac{1}{2}\frac{\partial b}{\partial x}b(t,x)(\Delta W(h))^2 + \frac{1}{2}\frac{\partial b}{\partial x}b(t,x)h + b(t,x)\Delta W(h) + \rho^3$$

$$= x + \left[\left(a - \frac{1}{2}\frac{\partial b}{\partial x}b\right)(t+h,X)\right]h + \frac{1}{2}\frac{\partial b}{\partial x}b(t+h,X)(\Delta W(h))^2 + b(t,x)\Delta W(h) + \rho^4, \tag{19}$$

where $|\boldsymbol{E}(\rho^3)| = O(h^2)$, $\boldsymbol{E}(\rho^3)^2 = O(h^3)$, $|\boldsymbol{E}(\rho^4)| = O(h^2)$, and $\boldsymbol{E}(\rho^4)^2 = O(h^3)$. However, the increments of the Wiener processes in (19) should be substituted by truncated random variables. Finally, by using Theorem 2.1, we prove the theorem. □

3 Numerical test

Example Consider the stochastic differential equation

$$\mathrm{d}X = tX(t)\mathrm{d}W(t), X(0) = X_0, \tag{20}$$

whose exact solution is

$$X(t) = X_0\exp\left(-\frac{1}{2}\int_0^t s^2\mathrm{d}s + \int_0^t s\mathrm{d}W(s)\right).$$

In application to (20), the Euler Method reads

$$X_{n+1} = X_n + t_n X_n \Delta W_n, \tag{21}$$

and the two full implicit methods we proposed have the forms

$$X_{n+1} = X_n - \frac{h}{4}\left(t_n + \frac{h}{2}\right)^2 (X_n + X_{n+1}) + \frac{1}{2}\left(t_n + \frac{h}{2}\right)(X_n + X_{n+1})\zeta_h \sqrt{h} \quad (22)$$

and

$$X_{n+1} = X_n - \frac{h}{2}(t_n + h)^2 X_{n+1} + t_n X_n \zeta_h \sqrt{h} + \frac{1}{2}(t_n + h)^2 X_{n+1}\zeta_h^2, \quad (23)$$

which are of the first mean-square order.

The numerical tests examine the behaviors of the numerical methods in two aspects: the sample trajectories produced by the numerical methods and the true solution shown in Fig. 1 (d) and the convergence rates of the numerical methods shown in Fig. 1(a), 1(b), and 1(c).

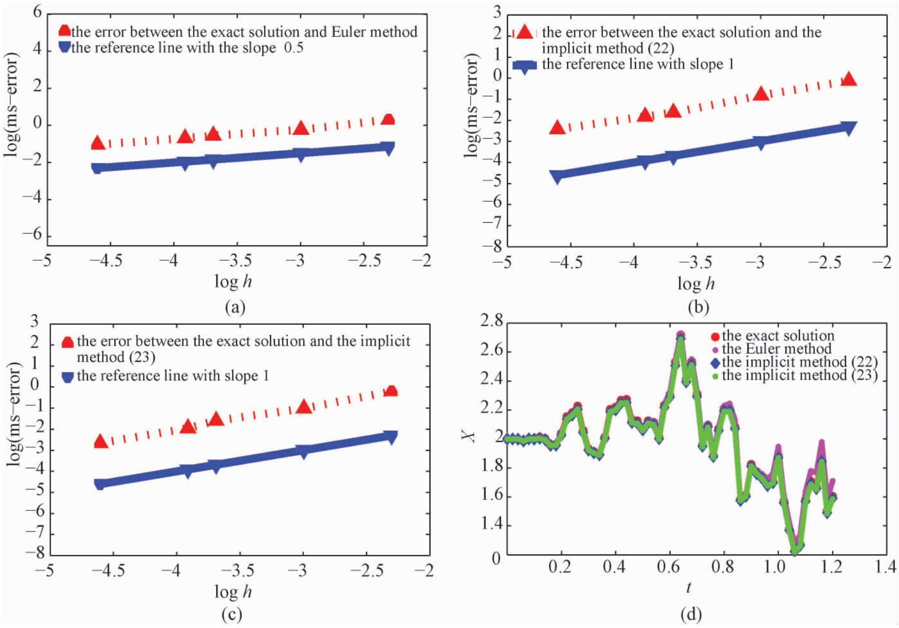

Fig. 1 Numerical test of the example

Figures 1(a), 1(b), and 1(c) show that, comparing with the reference line, the Euler method is of mean-square order 0.5 and the numerical methods (22) and (23) are of the first mean square order. In our experiments we take $t = 1$, $X_0 = 10$, and $h = [0.01, 0.02, 0.025, 0.05, 0.1]$.

Figure 1 (d) presents that the numerical approximations (22) and (23) are much closer to the exact solution than the Euler method for $X_0 = 2$ and $h = 0.04$ within the time interval $0 \leqslant t \leqslant 1.2$.

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