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Classification of the entangled states of $2\times L\times M\times N\times H$ *

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Abstract In this work we propose a practical entanglement classification scheme for pure states of $2\times L\times M\times N\times H$ under the stochastic local operation and classical communication (SLOCC), which generalizes the method explored in the entanglement classification of $2\times L\times M\times N$ to the five-partite system. The entangled states of $2\times L\times M\times N\times H$ system are first classified into different coarse-grained standard forms by using matrix decompositions, and then fine-grained identifications of two inequivalent entangled states with the same standard form are made by using the matrix realignment technique. As a practical example, the entanglement classes of the five-qubit system of $2\times 2\times 2\times 2\times 2$ are presented.

Key words quantum entanglement; SLOCC; nonlocal parameter

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$2\times L\times M\times N\times H$ 五体纠缠态分类

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摘 要 提出一个 $2\times L\times M\times N\times H$ 五体纠缠态基于随机定域操作和经典通信的实用分类方案. 该方案将 $2\times L\times M\times N$ 的纠缠态分类方法推广到五体系统. 首先利用矩阵分解将 $2\times L\times M\times N\times H$ 体系的纠缠态分类为不同的粗粒化的标准型,然后利用矩阵重排技术进一步对具有相同标准型的不等价纠缠态进行细粒化辨认. 最后给出一个 $2\times 2\times 2\times 2\times 2$ 五量子比特系统的分类举例.

关键词 量子纠缠; SLOCC 分类; 非定域参数

Entanglement has been an essential feature of quantum theory, and now is considered to be the key

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physical resource of quantum information sciences. Many nonclassical applications can only be implemented when entangled states are explored, e.g., quantum teleportation[1], dense coding[2-3], and some of the quantum cryptography protocols[4]. However, many superficially different quantum states may have actually the same function when being applied to carry out the quantum information tasks. It is known that, if two entangled states are interconnected by invertible local operators, i.e., equivalent under stochastic local operation and classical communication (SLOCC), then they would be both applicable for the same quantum information tasks. While there are only two SLOCC inequivalent tri-partite entanglement classes in three-qubit systems[5], the inequivalent classes turn to the infinite when the system consists of more than three partites.

The entanglement classification under SLOCC is generally a difficult task as the particles and dimensions of each partite grow, though it would be much easier when the entangled states have particular symmetries[6]. At present, nine inequivalent families of quantum systems for four-qubit states under SLOCC have been identified due to the symmetric property $SU(2) \otimes SU(2) \simeq SO(4)$[7]. Finer grained classifications could also be achieved with well constructed entangled measures[8-9]. Using the technique of coefficient matrix[10], 28 genuinely entangled families were found for the four-qubit system[11]. The rank of the coefficient matrix is useful in partitioning the entangled states into discrete entanglement families[12]. However as the dimensions and number of particles both grow, it provides a rather coarse grained classification[13]. New method for the entanglement classification of $2 \times L \times M \times N$ system has been proposed[14], and it takes full advantage of the classifications of $2 \times M \times N$ system[15-18]. The method not only provides an even finer classification for the system, but also is capable of determining the equivalency of two quantum states falling into the same entanglement family.

In this work, we generalize the method[14] to the case of five-partite system of $2 \times L \times M \times N \times H$. The five-partite system with one qubit is first partitioned into tri-partite in form of $2 \times (L \times M) \times (N \times H)$, and the standard forms of inequivalent entanglement classes of $2 \times (LM) \times (NH)$ behave as the entanglement families of $2 \times L \times M \times N \times H$. Then the matrix realignment is utilized to determine the equivalence of two entangled states and the connecting matrices between them within the same family.

1 Entanglement classification of pure system of $2 \times L \times M \times N \times H$

1.1 Representation of five-partite states

Every quantum state $|\psi>$ of five-partite system $2 \times L \times M \times N \times H$ may be formulated as

$$|\psi> = \sum_{i,m,n,l,h=1}^{2,M,N,L,H} \gamma_{ilmnh} |i,l,m,n,h>, \quad (1)$$

where $\gamma_{ilmnh} \in \mathbb{C}$ are coefficients of the state in representative bases. Therefore, the quantum state $|\psi>$ may also be represented as a high dimensional complex tensor ψ whose matrix elements are γ_{ilmnh}. In this form, the SLOCC equivalence of two quantum states ψ' and ψ [5] may be formulated as

$$\psi' = A^{(1)} \otimes A^{(2)} \otimes A^{(3)} \otimes A^{(4)} \otimes A^{(5)} \psi, \quad (2)$$

here $A^{(1)} \in \mathbb{C}^{2\times2}$, $A^{(2)} \in \mathbb{C}^{L\times L}$, $A^{(3)} \in \mathbb{C}^{M\times M}$, $A^{(4)} \in \mathbb{C}^{N\times N}$, and $A^{(5)} \in \mathbb{C}^{H\times H}$ are invertible matrices of 2×2, $L\times L$, $M\times M$, $N\times N$, and $H\times H$, respectively, which act on the corresponding particles.

For the sake of clarity, the quantum state ψ may also be formulated as $\psi \doteq \begin{pmatrix} \Gamma_1 \\ \Gamma_2 \end{pmatrix}$, and

$$\begin{pmatrix} \Gamma_1 \\ \Gamma_2 \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} \gamma_{11111} & \gamma_{11112} & \cdots & \gamma_{111NH} \\ \gamma_{11211} & \gamma_{11212} & \cdots & \gamma_{112NH} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{1LM11} & \gamma_{1LM12} & \cdots & \gamma_{1LMNH} \end{pmatrix} \\ \begin{pmatrix} \gamma_{21111} & \gamma_{21112} & \cdots & \gamma_{211NH} \\ \gamma_{21211} & \gamma_{21212} & \cdots & \gamma_{212NH} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{2LM11} & \gamma_{2LM12} & \cdots & \gamma_{2LMNH} \end{pmatrix} \end{pmatrix}, \quad (3)$$

which is obtained by grouping the particles as $2 \times (L$

$\times M) \times (N \times H)$. Here $\Gamma_i \in \mathbb{C}^{LM \times NH}$, i. e., complex matrices of LM columns and NH rows (we assume $LM \leqslant NH$ without loss of generalities).

1.2 Entanglement families of $2 \times L \times M \times N \times H$ system

It is easy to observe that the quantum state of tripartite system of $2 \times LM \times NH$ could also be represented in the same form as Eq. (3). Following the method[14], the SLOCC equivalence of two states ψ' and ψ in Eq. (2) transforms into the following form

$$\psi' = \boldsymbol{T} \otimes \boldsymbol{P} \otimes \boldsymbol{Q}^{\mathrm{T}} \psi, \tag{4}$$

and in the matrix pair representations, we have

$$\begin{pmatrix} \boldsymbol{\Gamma}'_1 \\ \boldsymbol{\Gamma}'_2 \end{pmatrix} = \boldsymbol{A}^{(1)} \begin{pmatrix} \boldsymbol{P} \boldsymbol{\Gamma}_1 \boldsymbol{Q} \\ \boldsymbol{P} \boldsymbol{\Gamma}_2 \boldsymbol{Q} \end{pmatrix}, \tag{5}$$

here $\boldsymbol{P} = \boldsymbol{A}^{(2)} \otimes \boldsymbol{A}^{(3)}, \boldsymbol{Q}^{\mathrm{T}} = \boldsymbol{A}^{(4)} \otimes \boldsymbol{A}^{(5)}$, T stands for matrix transposition, $\boldsymbol{A}^{(1)}$ acts on the two matrices $\boldsymbol{\Gamma}_{1,2}$, and $\boldsymbol{P}$ and $\boldsymbol{Q}$ act on the rows and columns of the $\boldsymbol{\Gamma}_{1,2}$ matrices, respectively. The SLOCC equivalence of two $2 \times L \times M \times N \times H$ quantum states in Eq. (5) has a form similar to the tripartite $2 \times LM \times NH$ pure state[16]. The differences lie in the fact that $\boldsymbol{P}$ and $\boldsymbol{Q}$ are not only invertible operators but also direct products of two invertible matrices, $\boldsymbol{A}^{(2)}$ and $\boldsymbol{A}^{(3)}$, $\boldsymbol{A}^{(4)}$ and $\boldsymbol{A}^{(5)}$.

As in Ref. [14], we have the following proposition.

Proposition 1.1 If two quantum states of $2 \times L \times M \times N \times H$ are SLOCC equivalent then their corresponding matrix-pairs have the same standard forms as those of $2 \times LM \times NH$ under the invertible operators $\boldsymbol{T} \in \mathbb{C}^{2\times 2}$, $\boldsymbol{P} \in \mathbb{C}^{LM \times LM}$, and $\boldsymbol{Q} \in \mathbb{C}^{NH \times NH}$.

This proposition serves as a necessary condition for the SLOCC equivalence of the entangled states of the $2 \times L \times M \times N \times H$ system.

The transforming matrices $\boldsymbol{T}_0$, $\boldsymbol{P}_0$, and $\boldsymbol{Q}_0$ for the standard form can be obtained. Generally the transformation matrices for the standard form are not unique. For example, if $\boldsymbol{T}_0$, $\boldsymbol{P}_0$, and $\boldsymbol{Q}_0$ are the matrices that transform ψ into its standard form, then the following matrices will do likewise,

$$\boldsymbol{T}_0 \otimes \boldsymbol{S} \boldsymbol{P}_0 \otimes (\boldsymbol{Q}_0 \boldsymbol{S}^{-1})^{\mathrm{T}} \psi = \begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{J} \end{pmatrix}, \tag{6}$$

where $\boldsymbol{S} \boldsymbol{J} \boldsymbol{S}^{-1} = \boldsymbol{J}$, i. e., $[\boldsymbol{S}, \boldsymbol{J}] = 0$. The nonuniqueness comes from the symmetries of the standard forms.

1.3 Entanglement classification of a $2 \times L \times M \times N \times H$ system

As the main result of the paper, we present the following theorem.

Theorem 1.1 Two $2 \times L \times M \times N \times H$ quantum states ψ and ψ' are SLOCC equivalent if and only if their corresponding matrix-pair representations have the same standard forms of $2 \times LM \times NH$ and the transformation matrices $\boldsymbol{P}$ and $\boldsymbol{Q}$ in Eq. (5) have the forms of direct products of two invertible matrices, i. e.,

$$\boldsymbol{P} = \boldsymbol{A}^{(2)} \otimes \boldsymbol{A}^{(3)} \text{ and } \boldsymbol{Q}^{\mathrm{T}} = \boldsymbol{A}^{(4)} \otimes \boldsymbol{A}^{(5)}.$$

Proof If two $2 \times L \times M \times N \times H$ quantum states ψ and ψ' are SLOCC equivalent, we have

$$\psi' = \boldsymbol{A}^{(1)} \otimes \boldsymbol{A}^{(2)} \otimes \boldsymbol{A}^{(3)} \otimes \boldsymbol{A}^{(4)} \otimes \boldsymbol{A}^{(5)} \psi, \tag{7}$$

here $\boldsymbol{A}^{(i)}$ is an invertible matrix, $i \in \{1,2,3,4,5\}$. According to Proposition 1.1, we have

$$\psi' = \boldsymbol{T} \otimes \boldsymbol{P} \otimes \boldsymbol{Q}^{\mathrm{T}}, \tag{8}$$

which means that ψ' and ψ have the same standard form of $2 \times LM \times NH$. Combining Eq. (7) and Eq. (8) yields

$$\boldsymbol{T}^{-1} \boldsymbol{A}^{(1)} \otimes (\boldsymbol{P}^{-1} (\boldsymbol{A}^{(2)} \otimes \boldsymbol{A}^{(3)})) \otimes ((\boldsymbol{Q}^{\mathrm{T}})^{-1} \boldsymbol{A}^{(4)} \otimes \boldsymbol{A}^{(5)}) \psi = \psi. \tag{9}$$

As the unit matrices $\boldsymbol{E} \otimes \boldsymbol{E} \otimes \boldsymbol{E}$ must be one of the operators which stabilizes the quantum state ψ in the matrix-pair form, $\boldsymbol{P}$ and $\boldsymbol{Q}^{\mathrm{T}}$ have the solutions of $\boldsymbol{P} = \boldsymbol{A}^{(2)} \otimes \boldsymbol{A}^{(3)}$ and $\boldsymbol{Q}^{\mathrm{T}} = \boldsymbol{A}^{(4)} \otimes \boldsymbol{A}^{(5)}$.

If the two quantum states have the same standard form, then we will have Eq. (8). And if further $\boldsymbol{P}$ and $\boldsymbol{Q}$ have the decomposions of $\boldsymbol{P} = \boldsymbol{P}_1 \otimes \boldsymbol{P}_2$ and $\boldsymbol{Q} = \boldsymbol{Q}_1 \otimes \boldsymbol{Q}_2$ where $\boldsymbol{P}_1 \in \mathbb{C}^{L \times L}$, $\boldsymbol{P}_2 \in \mathbb{C}^{M \times M}$ and $\boldsymbol{Q}_1 \in \mathbb{C}^{N \times N}$, $\boldsymbol{Q}_2 \in \mathbb{C}^{H \times H}$, $\boldsymbol{\psi}'$ and $\boldsymbol{\psi}$ are SLOCC equivalent entangled states of a $2 \times L \times M \times N \times H$ system. As matrices $\boldsymbol{P}$ and $\boldsymbol{Q}$ are invertible if and only if both $\boldsymbol{P}_1$, $\boldsymbol{P}_2$ and $\boldsymbol{Q}_1$, $\boldsymbol{Q}_2$ are invertible, thus

$$\psi' = \boldsymbol{T} \otimes (\boldsymbol{P}_1 \otimes \boldsymbol{P}_2) \otimes (\boldsymbol{Q}_1 \otimes \boldsymbol{Q}_2)^{\mathrm{T}} \psi. \tag{10}$$

Thus the classification procedure may be stated as follows. First, we construct the standard forms of the $2 \times LM \times NH$ system, which behave as the entanglement families of $2 \times L \times M \times N \times H$ and the transforming matrices $\boldsymbol{T}_0$, $\boldsymbol{P}_0$, and $\boldsymbol{Q}_0$ are also obtained. If two quantum states transform into different families, they are SLOCC inequivalent. Otherwise, the connecting matrices of $\boldsymbol{T}$, $\boldsymbol{P}$, and $\boldsymbol{Q}$ may be obtained. And we can determine whether such matrices have the direct product form or not using the matrix realignment technique[14]. Finally, Theorem 1.1 provides the complete entanglement classification for the two entangled states. In the following, we give detailed examples for $2 \times 2 \times 2 \times 2 \times 2$ quantum system as the application of our method.

2 Entanglement classification of $2 \times 2 \times 2 \times 2 \times 2$ system

There are totally 32 inequivalent families for the genuine $2 \times 2 \times 2 \times 2 \times 2$ entangled classes according to our method. The genuine entangled families of $2 \times 2 \times 2 \times 2 \times 2$ quantum states are listed as follows.

The $\mathcal{N}_f(22222) = 32$ families include:

two families from $2 \times 2 \times 2$ system (GHZ and W)

$$|\psi\rangle = |1(11)(11)\rangle + |2(22)(22)\rangle,$$
$$|\psi\rangle = |1(11)(11)\rangle + |1(22)(22)\rangle + |2(11)(22)\rangle,$$

two families from $2 \times 2 \times 3$ system

$$|\psi\rangle = |1(11)(11)\rangle + |1(12)(12)\rangle + |2(12)(21)\rangle,$$
$$|\psi\rangle = |1(11)(11)\rangle + |1(12)(12)\rangle + |2(11)(12)\rangle + |2(12)(21)\rangle,$$

one family from $2 \times 2 \times 4$ system

$$|\psi\rangle = |1(11)(11)\rangle + |1(12)(12)\rangle + |2(11)(21)\rangle + |2(12)(22)\rangle,$$

six families from $2 \times 3 \times 3$ system

$$|\psi\rangle = |1(11)(11)\rangle + |1(12)(12)\rangle + |2(21)(21)\rangle,$$
$$|\psi\rangle = |1(11)(11)\rangle + |1(12)(12)\rangle + |1(21)(21)\rangle + |2(11)(12)\rangle,$$
$$|\psi\rangle = |1(11)(11)\rangle + |1(12)(12)\rangle + |2(12)(12)\rangle + |2(21)(21)\rangle,$$
$$|\psi\rangle = |1(11)(11)\rangle + |1(12)(12)\rangle + |2(11)(12)\rangle + |2(21)(21)\rangle,$$
$$|\psi\rangle = |1(11)(11)\rangle + |1(12)(12)\rangle + |2(12)(21)\rangle + |2(21)(11)\rangle,$$
$$|\psi\rangle = |1(11)(11)\rangle + |1(12)(12)\rangle + |1(21)(21)\rangle + |2(11)(12)\rangle + |2(12)(21)\rangle.$$

five families from $2 \times 3 \times 4$ system

$$|\psi\rangle = |1(11)(11)\rangle + |1(12)(12)\rangle + |1(21)(21)\rangle + |2(21)(22)\rangle,$$
$$|\psi\rangle = |1(11)(11)\rangle + |1(12)(12)\rangle + |1(21)(21)\rangle + |2(11)(12)\rangle + |2(21)(22)\rangle,$$
$$|\psi\rangle = |1(11)(11)\rangle + |1(12)(12)\rangle + |1(21)(21)\rangle + |2(11)(11)\rangle + |2(21)(22)\rangle,$$
$$|\psi\rangle = |1(11)(11)\rangle + |1(12)(12)\rangle + |1(21)(21)\rangle + |2(12)(21)\rangle + |2(21)(22)\rangle,$$
$$|\psi\rangle = |1(11)(11)\rangle + |1(12)(12)\rangle + |1(21)(21)\rangle + |2(11)(12)\rangle + |2(12)(21)\rangle + |2(21)(22)\rangle.$$

The other 16 families come from the standard forms of a $2 \times 4 \times 4$ system. Among the 16 standard forms of $2 \times 4 \times 4$, there also exist the continuous entanglement families. That is, different entanglement families arise from the different values of the characterization parameters. We have proved that the standard forms, with the continuous parameters belonging to the same entanglement class of $2 \times 4 \times 4$ system, correspond to different entanglement families of $2 \times 2 \times 2 \times 2 \times 2$ system.

In addition, a necessary condition for the genuine entanglement of a $2 \times L \times M \times N \times H$ system is that all dimensions of the five particles shall be involved in the entanglement, requiring that $LM \leqslant 2NH$ (assuming the larger value of the dimensions to be LM). The scheme works better for higher dimensions, especially in the case of $LM = NH$.

3 Summaries

We have proposed a practical classification scheme for the entangled states of $2 \times L \times M \times N \times H$ pure system under SLOCC. By using the standard forms of $2 \times LM \times NH$, the entangled families of $2 \times L \times M \times N \times H$ are obtained. And the invertible local operators that connect two quantum states in the same family may also be constructed by using the matrix realignment technique. This provides a necessary and sufficient condition on the SLOCC equivalence of the two quantum states. As an application, detailed examples of the entanglement classification under SLOCC for five-qubit system is presented, which was not discussed systematically in the literature to the best of our knowledge.

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