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# Sharp bounds for generalized Hardy operator on product space*

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**Abstract** We characterize a sufficient and necessary condition which ensures that the generalized Hardy operator $\mathscr{U}_\psi f(x) = \int_0^1 \cdots \int_0^1 f(x_1 t_1, \cdots, x_n t_n) \psi(t_1, \cdots, t_n) \mathrm{d}t_1 \cdots \mathrm{d}t_n$ is bounded on RMO($\mathbb{R}^n$). The condition deeply depends on the nonnegative function $\psi$ defined on $[0,1] \times \cdots \times [0,1]$. Furthermore, the corresponding operator norm is worked out. In addition, we also extend the results to the high-dimensional product space.

**Key words** generalized Hardy operator; product space; RMO ($\mathbb{R}^n$)

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# 乘积空间上的广义哈代算子的有界性

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**摘　要** 研究乘积空间上的一类算子 $\mathscr{U}_\psi f(x) = \int_0^1 \cdots \int_0^1 f(x_1 t_1, \cdots, x_n t_n) \psi(t_1, \cdots, t_n) \mathrm{d}t_1 \cdots \mathrm{d}t_n$ 在 RMO($\mathbb{R}^n$)上有界的充分必要条件, 这个条件完全依赖于定义在$[0,1] \times \cdots \times [0,1]$上的非负函数 $\psi$. 还给出了 $\mathscr{U}_\psi$ 的算子范数. 此外, 还把这个结果推广到高维乘积空间.

**关键词** 广义哈代算子; 乘积空间; RMO($\mathbb{R}^n$)

In 1984, Carton-Lebrun and Fosset[1] defined the weighted Hardy-Littlewood average operator $U_\psi$

$$U_\psi f(x) = \int_0^1 f(xt) \psi(t) \mathrm{d}t, \tag{1}$$

where $\psi: [0,1] \to [0, \infty)$ is a function. Evidently the operator $U_\psi$ deeply depends on the nonnegative function $\psi$. For example, when $n = 1$ and $\psi(x) = 1$ for $x \in [0,1]$, the operator $H_\psi$ is just reduced to the

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classical Hardy operator

$$Hf(x) = \frac{1}{x}\int_0^x f(t)\,\mathrm{d}t, \qquad (2)$$

for $x \neq 0$. Consequently, $U_\psi$ is the more extensive Hardy operator and sometimes is called the generalized Hardy operator.

The classical Hardy operator $H$ is bounded on $L^p(\mathbb{R})$. That is, for $1 < p \leqslant \infty$,

$$\| Hf \|_{L^p(\mathbb{R})} \leqslant \frac{p}{p-1} \| f \|_{L^p(\mathbb{R})}$$

holds, where the constant $\frac{p}{p-1}$ is best possible.

In Ref. [2], Xiao considered the generalized Hardy operator $U_\psi$ and obtained the following theorem.

**Theorem A** Suppose that $\psi:[0,1] \rightarrow [0,\infty)$ is a nonnegative function and $p \in [1,\infty]$. Then the operator $U_\psi$ is bounded on $L^p(\mathbb{R}^n)$ if and only if

$$\int_0^1 t^{-\frac{1}{p}}\psi(t)\,\mathrm{d}t < \infty. \qquad (3)$$

Moreover, if the inequality (3) holds, then the operator norm of $U_\psi$ on $L^p(\mathbb{R}^n)$ is given by

$$\| U_\psi \|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} = \int_0^1 t^{-\frac{1}{p}}\psi(t)\,\mathrm{d}t; \qquad (4)$$

and $U_\psi: \mathrm{BMO}(\mathbb{R}^n) \rightarrow \mathrm{BMO}(\mathbb{R}^n)$ exists as a bounded operator if and only if

$$\int_0^1 \psi(t)\,\mathrm{d}t < \infty. \qquad (5)$$

Moreover, if the inequality (5) holds, then the operator norm of $U_\psi$ on $\mathrm{BMO}(\mathbb{R}^n)$ is given by

$$\| U_\psi \|_{\mathrm{BMO}(\mathbb{R}^n)\to \mathrm{BMO}(\mathbb{R}^n)} = \int_0^1 \psi(t)\,\mathrm{d}t. \qquad (6)$$

Recently, Chen et al.[3] studied the operator $\mathscr{U}_\psi$ defined as

$$\mathscr{U}_\psi f(x) = \int_0^1 \cdots \int_0^1 f(x_1 t_1, \cdots, x_n t_n)\,\psi(t_1, \cdots, t_n)\,\mathrm{d}t_1 \cdots \mathrm{d}t_n, \qquad (7)$$

where $f$ is defined on $\mathbb{R}^n$ and $\psi: [0,1]^n \rightarrow [0,\infty)$. $\mathscr{U}_\psi$ is an operator defined on the one dimensional product space. In Ref. [3], the following theorem was obtained.

**Theorem B** Suppose that $\psi:[0,1]^n \rightarrow [0,\infty)$ is a nonnegative function and $p \in [1,\infty]$. Then, the operator $\mathscr{U}_\psi$ is bounded on $L^p(\mathbb{R}^n)$ if and only if

$$\int_0^1 \cdots \int_0^1 (t_1 \cdots t_n)^{-\frac{1}{p}}\psi(t_1, \cdots, t_n)\,\mathrm{d}t < \infty. \qquad (8)$$

Moreover, if the inequality (8) holds, then the operator norm of $\mathscr{U}_\psi$ on $L^p(\mathbb{R}^n)$ is given by

$$\| \mathscr{U}_\psi \|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} = \int_0^1 \cdots \int_0^1 (t_1 \cdots t_n)^{-\frac{1}{p}}\psi(t_1, \cdots, t_n)\,\mathrm{d}t. \qquad (9)$$

Campanato space $\varepsilon^{\alpha,p}(\mathbb{R}^n)$ was first introduced by Campanato in Ref. [4]. The definition of Campanato space is as follows.

**Definition A** Let $-\infty < \alpha < \infty$ and $0 < p < \infty$. A locally integrable function $f$ is said to belong to Campanato space $\varepsilon^{\alpha,p}(\mathbb{R}^n)$ if there exists some constant $C > 0$ such that for any cube $Q \subset \mathbb{R}^n$ with all the sides parallel to the axes,

$$\frac{1}{|Q|^{\frac{\alpha}{n}}}\left(\frac{1}{|Q|}\int_Q |f(x) - f_Q|^p \,\mathrm{d}x\right)^{1/q} \leqslant C. \qquad (10)$$

The minimal constant $C$ is defined to be the $\varepsilon^{\alpha,p}(\mathbb{R}^n)$ norm of $f$ and denoted by $\| f \|_{\varepsilon^{\alpha,p}(\mathbb{R}^n)}$.

In Ref. [5], Zhao et al. studied the boundedness for $U_\psi$ on the space $\varepsilon^{\alpha,p}(\mathbb{R}^n)$. Theorem C was obtained.

**Theorem C** Suppose that $\psi:[0,1] \rightarrow [0,\infty)$ is a nonnegative function and $p \in [1,\infty)$, $-\frac{n}{p} \leqslant \alpha < 1$. Then $U_\psi$ is a bounded operator on $\varepsilon^{\alpha,p}(\mathbb{R}^n)$ if and only if

$$\int_0^1 t^\alpha \psi(t)\,\mathrm{d}t < \infty. \qquad (11)$$

Moreover,

$$\| U_\psi \|_{\varepsilon^{\alpha,p}(\mathbb{R}^n)\to \varepsilon^{\alpha,p}(\mathbb{R}^n)} = \int_0^1 t^\alpha \psi(t)\,\mathrm{d}t. \qquad (12)$$

Motivated by the previous studies[2-3,5], we devoted ourselves to investigating the boundedness of the operators $\mathscr{U}_\psi$ at the endpoint. A simple computation implies that $\mathscr{U}_\psi$ is not bounded on $\mathrm{BMO}(\mathbb{R}^n)$ and thus is not bounded on $\varepsilon^{\alpha,p}(\mathbb{R}^n)$, since $\varepsilon^{\alpha,p}(\mathbb{R}^n)$ equals to $\mathrm{BMO}(\mathbb{R}^n)$ as $\alpha = 0$. It is necessary for us to find some new spaces to replace

$BMO(\mathbb{R}^n)$ and $\varepsilon^{\alpha,p}(\mathbb{R}^n)$. In this work, we mainly consider this question and introduce two new spaces $RMO(\mathbb{R}^n)$ and $\mathscr{R}^{\alpha,p}(\mathbb{R}^n)$. We shall give their definitions as follows.

## 1 Some definitions

Before we put forward our main results, some useful definitions will be given. First we introduce the spaces $RMO(\mathbb{R}^n)$ and $\mathscr{R}^{\alpha,p}(\mathbb{R}^n)$ corresponding to $BMO(\mathbb{R}^n)$ and $\varepsilon^{\alpha,p}(\mathbb{R}^n)$, respectively.

**Definition 1.1** Let $f \in L_{loc}(\mathbb{R}^n)$. We say that $f \in RMO(\mathbb{R}^n)$ if and only if

$$\|f\|_{RMO(\mathbb{R}^n)} := \sup_{R \in \mathbb{R}^n} \frac{1}{|R|}\int_R |f(x) - f_R| \, dx < \infty, \tag{13}$$

where the supremum is taken over all rectangles with all the sides parallel to the axes and $f_R$ denotes the average of $f$ over $R$, i.e., $f_R = \frac{1}{|R|}\int_R f(x)\,dx$.

**Definition 1.2** Suppose that $f \in L_{loc}(\mathbb{R}^n)$, $-\infty < \alpha < \infty$ and $0 < p < \infty$. We say that $f \in \mathscr{R}^{\alpha,p}(\mathbb{R}^n)$ if and only if

$$\frac{1}{|R|^{\frac{\alpha}{n}}}\left(\frac{1}{|R|}\int_R |f(x) - f_R|^p dx\right)^{1/p} \leqslant C \tag{14}$$

holds for all rectangle $R \in \mathbb{R}^n$. The minimal constant $C$ is defined to be the $\mathscr{R}^{\alpha,p}(\mathbb{R}^n)$ norm of $f$ and denoted by $\|f\|_{\mathscr{R}^{\alpha,p}(\mathbb{R}^n)}$.

**Definition 1.3** Suppose that the measurable function $f$ is defined on $\mathbb{R}^{m_1} \times \cdots \times \mathbb{R}^{m_n}$ and $\varphi: [0,1]^n \to [0,\infty)$. The operator

$$\mathbb{U}_\varphi f(x) = \int_0^1 \cdots \int_0^1 f(t_1 x^1, \cdots, t_n x^n)\varphi(t_1, \cdots, t_n)\, dt_1 \cdots dt_n \tag{15}$$

is called the generalized Hardy operator on the high dimensional product space, where $x^i \in \mathbb{R}^{m_i}$ with $i = 1, 2, \cdots, n$.

Obviously, $\|\cdot\|_{RMO}$ forms a norm if we define RMO as the quotient space of all equivalent classes of functions whose difference is a constant. By Definition 1.1 and 1.2, it is not difficult for us to deduce that the space $BMO(\mathbb{R}^n)$ strictly contains $RMO(\mathbb{R}^n)$ and $RMO(\mathbb{R}^n) \supset L^\infty(\mathbb{R}^n)$.

Moreover, we conclude that the space $\mathscr{R}^{\alpha,p}(\mathbb{R}^n)$ is strictly contained in $\varepsilon^{\alpha,p}(\mathbb{R}^n)$ and equals to the space $RMO(\mathbb{R}^n)$ when $\alpha = 0$ as well.

## 2 Main results and their proofs

First, we study the boundedness of the operators $\mathscr{U}_\psi$ defined on the space RMO as in (13) and the space $\mathscr{R}^{\alpha,p}(\mathbb{R}^n)$ as in (14), and so does the operator $\mathbb{U}_\varphi$.

**Theorem 2.1** Let $\psi: [0,1]^n \to [0, +\infty)$ be a function. Then $\mathscr{U}_\psi: RMO(\mathbb{R}^n) \to RMO(\mathbb{R}^n)$ exists as a bounded operator if and only if

$$\int_0^1 \cdots \int_0^1 \psi(t)\,dt < \infty. \tag{16}$$

Moreover, when (16) holds, the operator norm of $\mathscr{U}_\psi$ on $RMO(\mathbb{R}^n)$ is given by

$$\|\mathscr{U}_\psi\|_{RMO(\mathbb{R}^n) \to RMO(\mathbb{R}^n)} = \int_0^1 \cdots \int_0^1 \psi(t)\,dt. \tag{17}$$

**Proof** In what follows, for each $t = (t_1, \cdots, t_n) \in \mathbb{R}^n$ with $t_i > 0$, $i = 1, \cdots, n$ and rectangle $R = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n] \in \mathbb{R}^n$, define $tR = [t_1 a_1, t_1 b_1] \times [t_2 a_2, t_2 b_2] \times \cdots \times [t_n a_n, t_n b_n]$. Assume that (16) holds. If $f \in RMO(\mathbb{R}^n)$, then, for any rectangle $R$, it follows from the Fubini's Theorem that

$$\begin{aligned}
(\mathscr{U}_\psi f)_R &= \frac{1}{|R|}\int_R (\mathscr{U}_\psi f)(y)\,dy \\
&= \frac{1}{|R|}\int_R \int_0^1 \cdots \int_0^1 f(t_1 y_1, \cdots, t_n y_n)\psi(t)\,dt\,dy \\
&= \int_0^1 \cdots \int_0^1 \frac{1}{|R|}\int_R f(t_1 y_1, \cdots, t_n y_n)\,dy\,\psi(t)\,dt \\
&= \int_0^1 \cdots \int_0^1 \frac{1}{|tR|}\int_{tR} f(z_1, \cdots, z_n)\,dy\,\psi(t)\,dt \\
&= \int_0^1 \cdots \int_0^1 f_{tR}\psi(t)\,dt,
\end{aligned} \tag{18}$$

where in the last equality we use the variable substitution $z_i = t_i y_i$ with $i = 1, 2, \cdots, n$. We conclude from the Minkowski's integral inequality that

$$\begin{aligned}
&\frac{1}{|R|}\int_R |(\mathscr{U}_\psi f)(y) - (\mathscr{U}_\psi f)_R|\,dy \\
&= \frac{1}{|R|}\int_R \left|(\mathscr{U}_\psi f)(y) - \int_0^1 \cdots \int_0^1 f_{tR}\psi(t)\,dt\right|dy
\end{aligned}$$

$$= \frac{1}{|R|}\int_R \left| \int_0^1 \cdots \int_0^1 \right.$$

$$\left. (f(t_1 y_1, \cdots, t_n y_n) - f_{tR})\psi(t)\,\mathrm{d}t \right| \mathrm{d}y \leqslant$$

$$\int_0^1 \cdots \int_0^1 \left( \frac{1}{|R|}\int_R |f(t_1 y_1, \cdots, t_n y_n) - f_{tR}|\,\mathrm{d}y \right)\psi(t)\,\mathrm{d}t =$$

$$\int_0^1 \cdots \int_0^1 \left( \frac{1}{|tR|}\int_{tR} |f(z_1, \cdots, z_n) - f_{tR}|\,\mathrm{d}z \right)\psi(t)\,\mathrm{d}t$$

$$\leqslant \int_0^1 \cdots \int_0^1 \psi(t)\,\mathrm{d}t \, \| f \|_{\mathrm{RMO}(\mathbb{R}^n)}. \qquad (19)$$

The inequality (19) shows that

$$\| \mathscr{U}_\psi f \|_{\mathrm{RMO}(\mathbb{R}^n)} \leqslant \int_0^1 \cdots \int_0^1 \psi(t)\,\mathrm{d}t \, \| f \|_{\mathrm{RMO}(\mathbb{R}^n)}.$$

Thus we have

$$\| \mathscr{U}_\psi \|_{\mathrm{RMO}(\mathbb{R}^n)} \leqslant \int_0^1 \cdots \int_0^1 \psi(t)\,\mathrm{d}t. \qquad (20)$$

Conversely, if $\mathscr{U}_\psi$ is bounded on $\mathrm{RMO}(\mathbb{R}^n)$, then we can choose

$$f_0(x) = \begin{cases} 1, & x \in \mathbb{R}_l^n, \\ -1, & x \in \mathbb{R}_r^n, \end{cases} \qquad (21)$$

where $\mathbb{R}_l^n$ and $\mathbb{R}_r^n$ denote the left and right halves of $\mathbb{R}^n$ respectively. In fact, $\mathbb{R}_l^n$ and $\mathbb{R}_r^n$ are separated by the hyperplane $x_1 = 0$, where $x_1$ is the first coordinate of $x \in \mathbb{R}^n$. At this point, a simple computation leads to

$$(\mathscr{U}_\psi f_0)(x) = \begin{cases} \int_0^1 \cdots \int_0^1 \psi(t)\,\mathrm{d}t, & x \in \mathbb{R}_l^n, \\ -\int_0^1 \cdots \int_0^1 \psi(t)\,\mathrm{d}t, & x \in \mathbb{R}_r^n. \end{cases} \qquad (22)$$

That is to say,

$$(\mathscr{U}_\psi f_0)(x) = f_0(x) \int_0^1 \cdots \int_0^1 \psi(t)\,\mathrm{d}t. \qquad (23)$$

By the definition of $f_0$, we clearly have $f_0 \in \mathrm{RMO}(\mathbb{R}^n)$ with $\| f \|_{\mathrm{RMO}(\mathbb{R}^n)} \neq 0$, and so does $\mathscr{U}_\psi f_0$ by (23). Obviously, (23) implies that

$$\| \mathscr{U}_\psi f_0 \|_{\mathrm{RMO}(\mathbb{R}^n)} = \| f_0 \|_{\mathrm{RMO}(\mathbb{R}^n)} \int_0^1 \cdots \int_0^1 \psi(t)\,\mathrm{d}t. \qquad (24)$$

Consequently, it follows from (24) that

$$\| \mathscr{U}_\psi \|_{\mathrm{RMO}(\mathbb{R}^n)} \geqslant \int_0^1 \cdots \int_0^1 \psi(t)\,\mathrm{d}t. \qquad (25)$$

Combining (20) with (25) yields the conclusion in (17). □

Using the almost same method, we obtain the following results.

**Corollary 2.1** Let $\varphi: [0,1]^n \to [0, +\infty)$ be a function. Then $\mathbb{U}_\varphi: \mathrm{RMO}(\mathbb{R}^{m_1} \times \cdots \times \mathbb{R}^{m_n}) \to \mathrm{RMO}(\mathbb{R}^{m_1} \times \cdots \times \mathbb{R}^{m_n})$ exists as a bounded operator if and only if

$$\int_0^1 \cdots \int_0^1 \varphi(t)\,\mathrm{d}t < \infty. \qquad (26)$$

Moreover, when (25) holds, the operator norm of $\mathbb{U}_\varphi$ on $\mathrm{RMO}(\mathbb{R}^{m_1} \times \cdots \times \mathbb{R}^{m_n})$ is given by

$$\| \mathbb{U}_\varphi \|_{\mathrm{RMO}(\mathbb{R}^{m_1} \times \cdots \times \mathbb{R}^{m_n}) \to \mathrm{RMO}(\mathbb{R}^{m_1} \times \cdots \times \mathbb{R}^{m_n})} = \int_0^1 \cdots \int_0^1 \varphi(t)\,\mathrm{d}t. \qquad (27)$$

Next we will consider the boundedness of the operators $\mathscr{U}_\psi$ on $\mathscr{R}^{\alpha,p}(\mathbb{R}^n)$.

**Theorem 2.2** Suppose that $-\frac{n}{p} < \alpha < \infty$ and $1 \leqslant p < \infty$. Let $\psi: [0,1]^n \to [0, +\infty)$ be a function. If

$$\int_0^1 \cdots \int_0^1 \prod_{i=1}^n t_i^{\alpha/n} \psi(t)\,\mathrm{d}t < \infty, \qquad (28)$$

then $\mathscr{U}_\psi$ is bounded from $\mathscr{R}^{\alpha,p}(\mathbb{R}^n)$ to $\mathscr{R}^{\alpha,p}(\mathbb{R}^n)$ and the following inequality

$$\| \mathscr{U}_\psi \|_{\mathscr{R}^{\alpha,p}(\mathbb{R}^n) \to \mathscr{R}^{\alpha,p}(\mathbb{R}^n)} \leqslant \int_0^1 \cdots \int_0^1 \prod_{i=1}^n t_i^{\alpha/n} \psi(t)\,\mathrm{d}t \qquad (29)$$

holds.

**Proof** Assume

$$\int_0^1 \cdots \int_0^1 \prod_{i=1}^n t_i^{\alpha/n} \psi(t)\,\mathrm{d}t < \infty$$

holds. Noting the equality (18), it follows from the Minkowski's integral inequality that

$$\frac{1}{|R|^{\frac{\alpha}{n}}} \left( \frac{1}{|R|}\int_R |(\mathscr{U}_\psi f)(y) - (\mathscr{U}_\psi f)_R|^p \mathrm{d}y \right)^{\frac{1}{p}}$$

$$= \frac{1}{|R|^{\frac{\alpha}{n}}} \left( \frac{1}{|R|}\int_R \right.$$

$$\left. \left| (\mathscr{U}_\psi f)(y) - \int_0^1 \cdots \int_0^1 f_{tR}\psi(t)\,\mathrm{d}t \right|^p \mathrm{d}y \right)^{\frac{1}{p}}$$

$$= \frac{1}{|R|^{\frac{\alpha}{n}}} \left( \frac{1}{|R|}\int_R \left| \int_0^1 \cdots \int_0^1 \right. \right.$$

$$(f(t_1y_1,\cdots,t_ny_n)-f_{tR})\psi(t)\,\mathrm{d}t|^p\mathrm{d}y)^{\frac{1}{p}}$$

$$\leqslant \frac{1}{|R|^{\frac{\alpha}{n}}}\int_0^1\cdots\int_0^1$$

$$\left(\frac{1}{|R|}\int_R|f(t_1y_1,\cdots,t_ny_n)-f_{tR}|^p\mathrm{d}y\right)^{\frac{1}{p}}\psi(t)\,\mathrm{d}t$$

$$=\int_0^1\cdots\int_0^1\frac{1}{|tR|^{\frac{\alpha}{n}}}\left(\frac{1}{|tR|}\int_{tR}\right.$$

$$\left.|f(z_1,\cdots,z_n)-f_{tR}|^p\mathrm{d}y\right)^{\frac{1}{p}}\prod_{i=1}^n t_i^{\alpha/n}\psi(t)\,\mathrm{d}t$$

$$\leqslant\int_0^1\cdots\int_0^1\prod_{i=1}^n t_i^{\alpha/n}\psi(t)\,\mathrm{d}t\,\|f\|_{\mathscr{B}^{\alpha,p}(\mathbb{R}^n)}. \quad (30)$$

The inequality (30) implies that

$$\|\mathscr{U}_\psi f\|_{\mathscr{B}^{\alpha,p}(\mathbb{R}^n)}\leqslant\int_0^1\cdots\int_0^1\prod_{i=1}^n t_i^{\alpha/n}\psi(t)\,\mathrm{d}t\,\|f\|_{\mathscr{B}^{\alpha,p}(\mathbb{R}^n)}.$$

Naturally we have

$$\|\mathscr{U}_\psi\|_{\mathscr{B}^{\alpha,p}(\mathbb{R}^n)\to\mathscr{B}^{\alpha,p}(\mathbb{R}^n)}\leqslant\int_0^1\cdots\int_0^1\prod_{i=1}^n t_i^{\alpha/n}\psi(t)\,\mathrm{d}t.$$

□

Now we formulate the similar conclusion on the high dimensional product space.

**Corollary 2.2** Suppose that $-\frac{n}{p}<\alpha<\infty$ and $1\leqslant p<\infty$. Let $\varphi:[0,1]^n\to[0,+\infty)$ be a function. if

$$\int_0^1\cdots\int_0^1\prod_{i=1}^n t_i^{\frac{\alpha m_i}{n}}\varphi(t)\,\mathrm{d}t<\infty, \quad (31)$$

then $\mathbb{U}_\varphi$ is bounded on $\mathscr{B}^{\alpha,p}(\mathbb{R}^{m_1}\times\cdots\times\mathbb{R}^{m_n})$, and the operator norm of $\mathbb{U}_\varphi$ is no more than

$$\int_0^1\cdots\int_0^1\prod_{i=1}^n t_i^{\frac{\alpha m_i}{n}}\varphi(t)\,\mathrm{d}t.$$

Using the same method as in the proof of Theorem B, the obtain the following corollary.

**Corollary 2.3** Suppose that $\varphi:[0,1]^n\to[0,\infty)$ is a nonnegative function and $p\in[1,\infty]$. Then the operator $\mathbb{U}_\varphi$ is bounded on $L^p(\mathbb{R}^{m_1}\times\cdots\times\mathbb{R}^{m_n})$ if and only if

$$\int_0^1\cdots\int_0^1\prod_{i=1}^n t_i^{-\frac{m_i}{p}}\varphi(t_1,\cdots,t_n)\,\mathrm{d}t<\infty. \quad (32)$$

Moreover, if the inequality (32) holds, then the operator norm of $\mathbb{U}_\varphi$ on $L^p(\mathbb{R}^{m_1}\times\cdots\times\mathbb{R}^{m_n})$ is given by

$$\|\mathbb{U}_\varphi\|_{L^p(\mathbb{R}^{m_1}\times\cdots\times\mathbb{R}^{m_n})\to L^p(\mathbb{R}^{m_1}\times\cdots\times\mathbb{R}^{m_n})}=\int_0^1\cdots\int_0^1\prod_{i=1}^n t_i^{-\frac{m_i}{p}}\varphi(t_1,\cdots,t_n)\,\mathrm{d}t. \quad (33)$$

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