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# Lie symmetry analysis and exact solutions of two-component Camassa-Holm equation *

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**Abstract** Using the Lie group analysis method, we study the two-component Camassa-Holm equation, which models shallow water waves moving over a linear shear flow. The similarity reductions and exact solutions for the equation are obtained. Then the power series solution are considered by using the power series method. Furthermore, the convergence of the power series solution to the equation is shown. The physical significance of the solutions is considered from the transformation group's point of view.

**Key words** Lie symmetry analysis; two-component Camassa-Holm equation; similarity reduction; exact solution

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# 二元 Camassa-Holm 方程的李对称分析和精确解

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**摘 要** 利用李群分析法研究二元 Camassa-Holm 方程,该方程以具有线性剪切流的浅水波为模型. 通过对称分析得到方程的相似约化和精确解,再用幂级数法获得方程的解. 证明了所得幂级数解的收敛性. 从变换群的角度考虑了方程所得解的物理意义.

**关键词** 李对称分析; 二元 Camassa-Holm 方程; 相似约化; 精确解

In this paper, we consider the following two-component Camassa-Holm equation

$$\begin{cases} u_t - u_{xxt} + 3uu_x - 2u_x u_{xx} - uu_{xxx} + \rho\rho_x = 0, \\ \rho_t + \rho_x u + \rho u_x = 0, \end{cases} \tag{1}$$

which was first derived as a bi-Hamiltonian model by Olver and Rosenau[1]. Eq. (1) includes both velocity and density variables in the dynamics of shallow water waves. In Ref. [2], Constantin and Ivanov derived (1) in the context of shallow water

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waves theory. The variable $u(x,t)$ describes the horizontal velocity of the fluid and the variable $\rho(x,t)$ is in connection with the horizontal deviation of the surface from the equilibrium, all measured in dimensionless units[2].

For $\rho \equiv 0$, (1) becomes the Camassa-Holm equation, modeling the unidirectional propagation of shallow water waves over a flat bottom. Here $u(x,t)$ stands for the fluid velocity at time $t$ in the spatial $x$-direction[3-8]. The Camassa-Holm equation is also a model for the propagation of axially symmetric waves in hyperelastic rods[9-10]. It has a bi-Hamiltonian structure[11-12] and is completely integrable[3,13]. It was claimed that the equation might be relevant to the modeling of tsunami[14] (see also the discussion in Ref. [15]).

The Cauchy problem and initial-boundary value problem for the Camassa-Holm equation have been studied extensively[16-20]. It has been shown that this equation is locally well-posed for some initial data. More interestingly, it has global strong solutions[16-17,21] and also finite time blow-up solutions[16-17,21-22]. For $\rho \neq 0$, the Cauchy problems of (1) have been discussed in Refs. [2,23].

For the sake of providing more information to understand the shallow water waves moving over a linear shear flow, we will investigate the vector fields, symmetry reductions, and exact solutions to (1) by the Lie symmetry analysis method.

It is known that the Lie symmetry analysis is a powerful and systematic method for dealing with partial differential equations (PDEs)[24-27]. Moreover, this method has had a profound impact on both pure and applied areas of mathematics, physics, and mechanics, etc. For the PDEs, admitting symmetry is one of the intrinsic properties of the equations. Based on the symmetries of a PDE, many other important properties of the equation such as integrability, conservation laws, reducing equations, and exact solutions can be considered successively[24-28].

The main purpose of this paper is to apply the Lie group analysis method for dealing with symmetries, symmetry reductions, and exact solutions to the two-component exact Camassa-Holm equation.

# 1 Lie symmetry analysis for the two-component Camassa-Holm equation

In this section, we perform Lie symmetry analysis for (1), and obtain its infinitesimal generators and commutation table of Lie algebra.

According to the method of determining the infinitesimal generator of PDEs, the infinitesimal generator of (1) can be written as

$$V = \xi(x,t,u,\rho)\frac{\partial}{\partial x} + \tau(x,t,u,\rho)\frac{\partial}{\partial t} + \phi(x,t,u,\rho)\frac{\partial}{\partial u} + \psi(x,t,u,\rho)\frac{\partial}{\partial \rho}, \quad (2)$$

where the coefficient functions $\xi(x,t,u,\rho)$, $\tau(x,t,u,\rho)$, $\phi(x,t,u,\rho)$, and $\psi(x,t,u,\rho)$ of the vector field are to be determined later.

If the vector field (2) generates a symmetry of (1), then $V$ must satisfy the Lie symmetry condition

$$\begin{cases} \mathrm{pr}^{(3)}V(\Delta_1)\mid_{\Delta_1=0} = 0, \\ \mathrm{pr}^{(1)}V(\Delta_2)\mid_{\Delta_2=0} = 0, \end{cases} \quad (3)$$

where $\mathrm{pr}^{(3)}V$ and $\mathrm{pr}^{(1)}V$ denote the 3rd and 1st prolongations of $V$, respectively, and

$$\Delta_1 = u_t - u_{xxt} + 3uu_x - 2u_xu_{xx} - uu_{xxx} + \rho\rho_x, \quad (4)$$

$$\Delta_2 = \rho_t + \rho_x u + \rho u_x, \quad (5)$$

for Eq. (1). That is

$$\begin{cases} \mathrm{pr}^{(3)}V = \phi\frac{\partial}{\partial u} + \psi\frac{\partial}{\partial \rho} + \phi^x\frac{\partial}{\partial u_x} + \psi^x\frac{\partial}{\partial \rho_x} + \\ \quad \phi^t\frac{\partial}{\partial u_t} + \phi^{xx}\frac{\partial}{\partial u_{xx}} + \phi^{xxx}\frac{\partial}{\partial u_{xxx}} + \phi^{xxt}\frac{\partial}{\partial u_{xxt}}, \\ \mathrm{pr}^{(1)}V = \phi\frac{\partial}{\partial u} + \psi\frac{\partial}{\partial \rho} + \phi^x\frac{\partial}{\partial u_x} + \psi^x\frac{\partial}{\partial \rho_x} + \psi^t\frac{\partial}{\partial \rho_t}, \end{cases} \quad (6)$$

where $\phi^x, \psi^x, \phi^t, \psi^t, \phi^{xx}, \phi^{xxx}$, and $\phi^{xxt}$ are the coefficients of $\mathrm{pr}^{(3)}V$ and $\mathrm{pr}^{(1)}V$. Furthermore, we have

$$\phi^x = D_x\phi - u_xD_x\xi - u_tD_x\tau, \quad (7)$$

$$\phi^t = D_t\phi - u_xD_t\xi - u_tD_t\tau, \quad (8)$$

$$\psi^x = D_x\psi - \rho_xD_x\xi - \rho_tD_x\tau, \quad (9)$$

$$\psi^t = D_t\psi - \rho_xD_t\xi - \rho_tD_t\tau, \quad (10)$$

$$\phi^{xx} = D_x^2(\phi - \xi u_x - \tau u_t) + \xi u_{xxx} + \tau u_{xxt}, \tag{11}$$

$$\phi^{xxt} = D_t D_x^2(\phi - \xi u_x - \tau u_t) + \xi u_{xxxt} + \tau u_{xxtt}, \tag{12}$$

$$\phi^{xxx} = D_x^3(\phi - \xi u_x - \tau u_t) + \xi u_{xxxx} + \tau u_{xxxt}, \tag{13}$$

where $D_x$ and $D_t$ are the total derivatives with respect to $x$ and $t$, respectively.

Substituting (7) – (13) into (6), in terms of the Lie symmetry analysis method, we obtain the following equations for the symmetry group of (1)

$$\begin{cases} \xi_x = \xi_t = \xi_u = \xi_\rho = 0, \\ \tau_x = \tau_u = \tau_\rho = 0, \tau_{t,t} = 0, \\ \phi = -u\tau_t, \\ \psi = -\rho\tau_t. \end{cases} \tag{14}$$

Solving (14), we obtain

$$\xi = c_1,\ \tau = c_2 t + c_3,$$
$$\phi = -c_2 u,\ \psi = -c_2\rho, \tag{15}$$

where $c_1, c_2$, and $c_3$ are arbitrary constants.

Hence the Lie algebra of infinitesimal symmetries of (1) is spanned by the following vector fields

$$V_1 = \frac{\partial}{\partial x},\ V_2 = t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u} - \rho\frac{\partial}{\partial \rho},$$
$$V_3 = \frac{\partial}{\partial t}. \tag{16}$$

It is easy to verify that $\{V_1, V_2, V_3\}$ is closed under the Lie bracket. In fact, we have

$$[V_1, V_1] = [V_2, V_2] = [V_3, V_3] = 0,$$
$$[V_1, V_2] = -[V_2, V_1] = [V_1, V_3] = -[V_3, V_1] = 0,$$
$$[V_2, V_3] = -[V_3, V_2] = -V_3. \tag{17}$$

## 2 Symmetry groups of the two-component Camassa-Holm equation

In section 1, we have obtained the infinitesimal symmetries of (1). Furthermore, for (1), the one-parameter groups $G_i$ generated by $V_1$, $V_2$, and $V_3$ are given as follows:

$$G_1: (x, t, u, \rho) \rightarrow (x + \epsilon, t, u, \rho),$$
$$G_2: (x, t, u, \rho) \rightarrow (x, te^{\epsilon}, ue^{-\epsilon}, \rho e^{-\epsilon}),$$
$$G_3: (x, t, u, \rho) \rightarrow (x, t + \epsilon, u, \rho),$$

where $\epsilon$ is any real number. We note that $G_1$ is a space translation; $G_3$ is a time translation; and $G_2$ is a genuinely local group of transformation. They are very important in our study of the exact solutions of PDEs.

Consequently, if $u = f(x, t)$ and $\rho = g(x, t)$ is a solution of (1), then $u_{(i)}$ and $\rho_{(i)}$ $(i = 1, 2, 3)$ given as follows are solutions of (1) as well

$$u_{(1)} = f(x - \epsilon, t),\ \rho_{(1)} = g(x - \epsilon, t), \tag{18}$$
$$u_{(2)} = e^{-\epsilon} f(x, te^{-\epsilon}),\ \rho_{(2)} = e^{-\epsilon} g(x, te^{-\epsilon}), \tag{19}$$
$$u_{(3)} = f(x, t - \epsilon),\ \rho_{(3)} = g(x, t - \epsilon). \tag{20}$$

where $\epsilon$ is any real number.

## 3 Symmetry reductions and exact solutions of the two-component Camassa-Holm equation

Now we deal with the exact solutions for (1) based on the symmetry analysis. To do this, linear combinations of infinitesimals are considered and their corresponding invariants are determined.

(i) For the generator $V_1 = \frac{\partial}{\partial x}$, we have the similarity variables

$$\xi = t,\ \omega = u,\ \nu = \rho,$$

and the group – invariant solution is $\omega = f(\xi)$, $\nu = g(\xi)$, that is,

$$u = f(t),\ \rho = g(t). \tag{21}$$

Substituting Eq. (21) into Eq. (1), we obtain the reduction equation

$$\begin{cases} f' = 0, \\ g' = 0, \end{cases} \tag{22}$$

where $f' = \frac{df}{d\xi}$, $g' = \frac{dg}{d\xi}$. Therefore, (1) has a solution $u = c_1$, $\rho = c_2$, where $c_1$, $c_2$ are arbitrary constants. Obviously, the solution is not meaningful.

(ii) For the generator $V_2 = t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u} - \rho\frac{\partial}{\partial \rho}$, we have the similarity variables

$$\xi = x,\ \omega = ut,\ \nu = \rho t,$$

and the group-invariant solution is $\omega = f(\xi)$, $\nu = g(\xi)$, that is,

$$u = f(x)t^{-1},\ \rho = g(x)t^{-1}. \tag{23}$$

Substituting (23) into (1), we obtain the reduction equation

$$\begin{cases} -f + 3ff' + f'' - 2f'f'' - ff^{(3)} + gg' = 0, \\ -g + gf' + g'f = 0, \end{cases} \tag{24}$$

where $f' = \frac{\mathrm{d}f}{\mathrm{d}\xi}$, $g' = \frac{\mathrm{d}g}{\mathrm{d}\xi}$.

(iii) For the generator $V_1 + V_2 = \frac{\partial}{\partial x} + t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u} - \rho\frac{\partial}{\partial \rho}$, we have the similarity variables

$$\xi = te^{-x},\ \omega = ue^{x},\ \nu = \rho e^{x},$$

and the group-invariant solution is $\omega = f(\xi)$, $\nu = g(\xi)$, that is,

$$u = e^{-x}f(te^{-x}),\ \rho = e^{-x}g(te^{-x}). \tag{25}$$

Substituting (25) into (1), we obtain the following reduction equation

$$\begin{cases} -3f' + 12\xi ff' + 6\xi^2 f'^2 - 5\xi f'' + 8\xi^2 ff'' + \\ \quad 2\xi^3 f'f'' - \xi^2 f^{(3)} + \xi^3 ff^{(3)} - g^2 - \xi gg' = 0, \\ -2gf + g' - \xi gf' - \xi g'f = 0, \end{cases} \tag{26}$$

where $f' = \frac{\mathrm{d}f}{\mathrm{d}\xi}$, $g' = \frac{\mathrm{d}g}{\mathrm{d}\xi}$.

(iv) For the infinitesimal generator $cV_1 + V_3 = c\frac{\partial}{\partial x} + \frac{\partial}{\partial t}$, we have the similarity variables

$$\xi = x - ct,\ \omega = u,\ \nu = \rho,$$

and the group – invariant solution is $\omega = f(\xi)$, $\nu = g(\xi)$, that is,

$$u = f(x - ct),\ \rho = g(x - ct). \tag{27}$$

Substituting (27) into (1), we obtain the reduction equation

$$\begin{cases} (3f - c)f' - 2f'f'' + (c - f)f^{(3)} + gg' = 0, \\ (f - c)g' + gf' = 0, \end{cases} \tag{28}$$

where $f' = \frac{\mathrm{d}f}{\mathrm{d}\xi}$, $g' = \frac{\mathrm{d}g}{\mathrm{d}\xi}$.

**Remark 3.1** Noting that the reduced equations such as (24) and (26) are all higher-order nonlinear or nonautonomous ODEs, we will deal with such equations in the next section.

# 4 The power series solutions

In section 3, we have obtained the reduced equations by using Lie symmetry reductions. In this section, we will treat the nonlinear ODEs (24), (26), and (28). The power series can be used to solve differential equations, including many complicated differential equations with nonconstant coefficients[29]. Now we consider the power series solutions to the reduced equations.

## 4.1 The power series solutions to Eq. (24)

Now, we seek a solution of (24) in the form of a power series

$$f(\xi) = \sum_{n=0}^{\infty} p_n \xi^n,\ g(\xi) = \sum_{n=0}^{\infty} q_n \xi^n, \tag{29}$$

where the coefficients $p_n$ and $q_n$ are all constants to be determined.

Substituting (29) into (24), we have

$$\begin{gathered} -\sum_{n=0}^{\infty} p_n\xi^n + 3\sum_{n=0}^{\infty}\sum_{k=0}^{n}(n+1-k)p_k p_{n+1-k}\xi^n + \\ \sum_{n=0}^{\infty}(n+1)(n+2)p_{n+2}\xi^n - 2\sum_{n=0}^{\infty}\sum_{k=0}^{n}(k+1) \\ (n+1-k)(n+2-k)p_{k+1}p_{n+2-k}\xi^n - \sum_{n=0}^{\infty}\sum_{k=0}^{n} \\ (n+1-k)(n+2-k)(n+3-k)p_k p_{n+3-k}\xi^n + \\ \sum_{n=0}^{\infty}\sum_{k=0}^{n}(n+1-k)q_k q_{n+1-k}\xi^n = 0, \\ -\sum_{n=0}^{\infty} q_n\xi^n + \sum_{n=0}^{\infty}\sum_{k=0}^{n}(n+1-k)q_k p_{n+1-k}\xi^n + \\ \sum_{n=0}^{\infty}\sum_{k=0}^{n}(n+1-k)p_k q_{n+1-k}\xi^n = 0. \end{gathered} \tag{30}$$

From (30), comparing coefficients, we obtain the recursion formula

$$\begin{gathered} p_{n+3} = \frac{1}{(n+1)(n+2)(n+3)p_0}\{-p_n + (n+1) \\ [3p_0p_{n+1} + (n+2)(1-2p_1)p_{n+2} + q_0q_{n+1}] + \\ \sum_{k=1}^{n}(n+1-k)[3p_kp_{n+1-k} + q_kq_{n+1-k} - \\ (n+2-k)(2(k+1)p_{k+1}p_{n+2-k} + \\ (n+3-k)p_kp_{n+3-k})]\}, \\ q_{n+1} = \frac{1}{(n+1)p_0}[q_n - (n+1)q_0p_{n+1} - \\ \sum_{k=1}^{n}(n+1-k)(q_kp_{n+1-k} + p_kq_{n+1-k})], \end{gathered} \tag{31}$$

for all $n = 0, 1, 2, \cdots$.

Thus, for arbitrarily chosen constants $p_0 \neq 0$, $p_1, p_2$, and $q_0$, from (31), we obtain

$$p_3 = \frac{p_0(-1 + 3p_1) + 2p_2(1 - 2p_1) + q_0q_1}{6p_0},$$

$$q_1 = \frac{q_0(1-p_1)}{p_0}. \tag{32}$$

then, we have

$$p_4 = \frac{p_1(3p_1-1)+2p_2(3p_0-4p_2)}{24p_0} + \frac{6p_3(1-3p_1)+2q_0q_2+q_1^2}{24p_0},$$

$$q_2 = \frac{q_1-2q_0p_2-2p_1q_1}{2p_0}, \tag{33}$$

and so on.

Thus, the other terms of the sequence $\{p_n\}_{n=0}^{\infty}$ and $\{q_n\}_{n=0}^{\infty}$ can be determined successively from (31) in a unique manner. This implies that, for (24), there exists a power series solution (29) with the coefficients given in (31). Furthermore, it is easy to prove the convergence of the power series (29) with the coefficients given in (31). As an example, we consider the convergence of the power series solution (29) of (24). From (31), we have

$$|p_{n+3}| \leqslant M[|p_n|+|p_{n+1}|+|p_{n+2}|+|q_{n+1}|+ \sum_{k=1}^{n}(|p_k||p_{n+1-k}|+|q_k||q_{n+1-k}|+ |p_{k+1}||p_{n+2-k}|+|p_k||p_{n+3-k}|)],$$
$$n=0,1,2,\cdots,$$

where $M = \max\left\{\frac{1}{|p_0|}, 1, \frac{|1-2p_1|}{|p_0|}, \frac{|q_0|}{|p_0|}\right\}$.

Similarly, from (31), we have

$$|q_{n+1}| \leqslant N[|q_n|+|p_{n+1}|+\sum_{k=1}^{n}(|q_k||p_{n+1-k}|+ |p_k||q_{n+1-k}|)],$$
$$n=0,1,2,\cdots,$$

where $N = \max\left\{\frac{1}{|p_0|}, \frac{|q_0|}{|p_0|}\right\}$.

Now, we define two power series $R=R(\xi)=\sum_{n=0}^{\infty} r_n\xi^n$ and $S=S(\xi)=\sum_{n=0}^{\infty} s_n\xi^n$, by taking

$$r_i = |p_i|,\ s_j = |q_j|,\ i=0,1,2,\ j=0,$$

and

$$r_{n+3} = M[|r_n|+|r_{n+1}|+|r_{n+2}|+|s_{n+1}|+ \sum_{k=1}^{n}(|r_k||r_{n+1-k}|+|s_k||s_{n+1-k}|+ |r_{k+1}||r_{n+2-k}|+|r_k||r_{n+3-k}|)],$$

$$s_{n+1} = N[|s_n|+|r_{n+1}|+\sum_{k=1}^{n}(|s_k||r_{n+1-k}|+ |r_k||s_{n+1-k}|)],$$

where $n=0,1,2,\cdots$. Then, it is easily seen that

$$|p_n| \leqslant r_n,\ |q_n| \leqslant s_n,\ n=0,1,2,\cdots.$$

In other words, the two series $R=R(\xi)=\sum_{n=0}^{\infty} r_n\xi^n$ and $S=S(\xi)=\sum_{n=0}^{\infty} s_n\xi^n$ are majorant series of (29), respectively. Next, we show that the series $R=R(\xi)$ and $S=S(\xi)$ have positive radius of convergence. Indeed, by formal calculation, we have

$$\begin{aligned} R(\xi) &= r_0+r_1\xi+r_2\xi^2+\sum_{n=0}^{\infty} r_{n+3}\xi^{n+3} \\ &= r_0+r_1\xi+r_2\xi^2+M[\sum_{n=0}^{\infty} r_n\xi^{n+3}+ \sum_{n=0}^{\infty} r_{n+1}\xi^{n+3}+\sum_{n=0}^{\infty} r_{n+2}\xi^{n+3}+\sum_{n=0}^{\infty} s_{n+1}\xi^{n+3}+\sum_{n=0}^{\infty}\sum_{k=1}^{n} r_kr_{n+1-k}\xi^{n+3}+\sum_{n=0}^{\infty}\sum_{k=1}^{n} s_ks_{n+1-k}\xi^{n+3}+\sum_{n=0}^{\infty}\sum_{k=1}^{n} r_{k+1}r_{n+2-k}\xi^{n+3}+ \sum_{n=0}^{\infty}\sum_{k=1}^{n} r_kr_{n+3-k}\xi^{n+3}] \\ &= r_0+r_1\xi+r_2\xi^2+M[\xi^3R+(\xi^2-r_1\xi- r_0\xi^2+\xi^2R-\xi^2r_2)(R-r_0)+2R^2- 3r_0R+r_0^2+(\xi-r_1\xi-r_0)(R-r_0- r_1\xi)+(\xi^2-s_0\xi^2+\xi^2S)(S-s_0)-\xi r_1R], \end{aligned}$$

and

$$S(\xi) = s_0+N[\xi S+(1+2S-2s_0)(R-r_0)].$$

Consider now the implicit functional system with respect to the independent variable $\xi$,

$$\begin{aligned} F(\xi,R,S) &= R-r_0-r_1\xi-r_2\xi^2-M[\xi^3R+ (\xi^2-r_1\xi-r_0\xi^2+\xi^2R-\xi^2r_2) (R-r_0)+(\xi-r_1\xi-r_0)(R-r_0-r_1\xi)+ (\xi^2-s_0\xi^2+\xi^2S) (S-s_0)-\xi r_1R+2R^2-3r_0R+r_0^2]=0, \\ G(\xi,R,S) &= S-s_0-N[\xi S+ (1+2S-2s_0)(R-r_0)]=0. \end{aligned}$$

Since $F, G$ are analytic in the neighborhood of $(0, r_0, s_0)$, $F(0,r_0,s_0)=0, G(0,r_0,s_0)=0$.

Furthermore, the Jacobian determinant

$$\left.\frac{\partial(F,G)}{\partial(R,S)}\right|_{(0,r_0,s_0)} = 1 \neq 0,$$

if we choose the parameters $r_0=|p_0|$ and $s_0=|q_0|$

properly. By the implicit function theorem[30], we see that $R = R(\xi)$ and $S = S(\xi)$ are analytic in a neighborhood of the point $(0, r_0, s_0)$ and with the positive radius. This implies that the two power series (29) converge in a neighborhood of the point $(0, r_0, s_0)$.

The power series solution of (1) can be written as

$$\begin{aligned} u(x,t) &= p_0 t^{-1} + p_1 x t^{-1} + p_2 x^2 t^{-1} + p_3 x^3 t^{-1} + \sum_{n=1}^{\infty} p_{n+3} x^{n+3} t^{-1} \\ &= p_0 t^{-1} + p_1 x t^{-1} + p_2 x^2 t^{-1} + \frac{p_0(-1+3p_1) + 2p_2(1-2p_1) + q_0 q_1}{6p_0} x^3 t^{-1} + \\ &\quad \sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)(n+3)p_0} \{ -p_n + (n+1)[3p_0 p_{n+1} + (n+2)(1-2p_1)p_{n+2} + q_0 q_{n+1}] + \sum_{k=1}^{n}(n+1-k)[3p_k p_{n+1-k} + \\ &\quad q_k q_{n+1-k} - (n+2-k)(2(k+1)p_{k+1} p_{n+2-k} + (n+3-k)p_k p_{n+3-k})]\} x^{n+3} t^{-1}, \\ \rho(x,t) &= q_0 t^{-1} + q_1 x t^{-1} + \sum_{n=1}^{\infty} q_{n+1} x^{n+1} t^{-1} \\ &= q_0 t^{-1} + \frac{q_0(1-p_1)}{p_0} x t^{-1} + \sum_{n=1}^{\infty} \frac{1}{(n+1)p_0} [q_n - (n+1)q_0 p_{n+1} - \sum_{k=1}^{n}(n+1-k) \\ &\quad (q_k p_{n+1-k} + p_k q_{n+1-k})] x^{n+1} t^{-1}, \end{aligned} \tag{34}$$

where $p_0 \neq 0, p_1, p_2$, and $q_0$ are arbitrary constants. The other coefficients $p_n (n \geqslant 3)$ and $q_n (n \geqslant 1)$ can be determined successively from (31).

In physical applications, it will be convenient to write the solution of (1) in the approximate form

$$\begin{aligned} u(x,t) &= p_0 t^{-1} + p_1 x t^{-1} + p_2 x^2 t^{-1} + \frac{p_0(-1+3p_1) + 2p_2(1-2p_1) + q_0 q_1}{6p_0} x^3 t^{-1} + \\ &\quad \frac{p_1(3p_1 - 1) + 2p_2(3p_0 - 4p_2)}{24p_0} x^4 t^{-1} + \frac{6p_3(1-3p_1) + 2q_0 q_2 + q_1^2}{24p_0} x^4 t^{-1} + \cdots, \\ \rho(x,t) &= q_0 t^{-1} + \frac{q_0(1-p_1)}{p_0} x t^{-1} + \frac{q_1 - 2q_0 p_2 - 2p_1 q_1}{2p_0} x^2 t^{-1} + \cdots, \end{aligned}$$

in terms of the above computation.

## 4.2 The power series solutions to (26)

Now, we seek a solution of (26) in the form of the power series (29). Substituting (29) into (26) and comparing the coefficients, we obtain

$$p_1 = -\frac{q_0^2}{3}, \quad q_1 = 2q_0 p_0,$$

$$p_2 = \frac{3(-q_0 q_1 + 4p_0 p_1)}{16},$$

$$p_3 = \frac{2(-2q_0 q_2 - q_1^2 + 20p_0 p_2 + 9p_1^2)}{45}. \tag{35}$$

Generally, for $n \geqslant 0$, we have

$$\begin{aligned} p_{n+4} &= \frac{1}{(n+4)^2(n+6)} \{ \sum_{k=0}^{n} [-(n-k+4)q_k q_{n-k+3} + (n-k+3)(12 + (n-k+2)(n-k+9))p_k p_{n-k+3} + 2(k+1)(n-k+2)(n-k+4)p_{k+1} p_{n-k+2}] - \\ &\quad 3q_{n+1} q_2 - 2q_{n+2} q_1 - q_{n+3} q_0 + 40 p_{n+1} p_2 + 6(n+4)p_{n+2} p_1 \}, \\ q_{n+2} &= \frac{1}{(n+2)} \{ \sum_{k=0}^{n} [(n-k+3)q_k p_{n-k+1} + (n-k+1)p_k q_{n-k+1}] + 2q_{n+1} p_0 \}. \end{aligned} \tag{36}$$

Thus, for arbitrarily chosen constants $p_0$ and $q_0$, from (36) we have

$$p_4 = \frac{-5q_0 q_3 + 90p_0 p_3 + 80p_1 p_2 - 5q_1 q_2}{96},$$

$$q_2 = \frac{3}{2}(q_0 p_1 + q_1 p_0), \tag{37}$$

and so on.

Thus, for arbitrary chosen constants $p_0$ and $q_0$, the other terms of the sequence $\{p_n\}_{n=0}^{\infty}$ and $\{q_n\}_{n=0}^{\infty}$ can be determined successively from (35) and (36) in a unique manner. This implies that, for (26), there exists a power series solution (29) with the coefficients given by (35) and (36).

## 4.3 The power series solutions to (28)

Similarly, we seek a solution of (28) in the form of the power series (29). Substituting (29) into (28) and comparing the coefficients, we obtain recursion formula

$$p_{n+3} = \frac{1}{c(n+1)(n+2)(n+3)} \{ c(n+1)p_{n+1} + \sum_{k=0}^{n}(n+1-k)[-3p_k p_{n+1-k} - q_k q_{n+1-k} + 2(k+1)(n+2-k)p_{k+1} p_{n+2-k} +$$

$$(n+3-k)(n+2-k)p_k p_{n+3-k}]\},$$

$$q_{n+1} = \frac{1}{c(n+1)}\Big[\sum_{k=0}^{n}(n+1-k)(q_k p_{n+1-k} + p_k q_{n+1-k})\Big], \tag{38}$$

for all $n = 0,1,2,\cdots$.

Thus, for arbitrarily chosen constants $p_0 \neq 0$, $p_1$, $p_2$, and $q_0$, from (38), we obtain

$$p_3 = \frac{cp_1 - 3p_0p_1 + 4p_1p_2 + 6p_0p_3 - q_0q_1}{6c},$$

$$q_1 = \frac{q_0p_1 + q_1p_0}{c}, \tag{39}$$

and so on.

Thus, for arbitrary chosen constants $c \neq 0$, $p_0 \neq 0$, $p_1$, $p_2$, and $q_0$, the other terms of the sequence $\{p_n\}_{n=0}^{\infty}$ and $\{q_n\}_{n=0}^{\infty}$ can be determined successively from (38) in a unique manner. This implies that, for (28), there exists a power series solution (29) with the coefficients given by (38).

The approximate solution of (1) can be obtained in terms of the above computation. The details are omitted here.

**Remark 4.1** The proofs for the convergence of the power series solutions to (26) and (28) are similar to the one for (24). The details are omitted here. We reiterate that the power series solutions which have been obtained in this paper are approximate solutions. Furthermore, such power series solutions converge quickly. So it is convenient for computations in both theory and application.

# 5 Conclusions

In this paper, the exact solutions of the two-component Camassa-Holm equation have been studied by using Lie symmetry analysis and the power series method. The similarity reductions and exact solutions are given for the first time. These similarity solutions possess significant features in physical systems. The results of this paper show that the Lie symmetry analysis is a very powerful method.

## References


[1] Olver P J, Rosenau P. Tri-Hamiltonian duality between solitons and solitary-wave solutions having compact support [J]. Physical Review E, 1996, 53: 1 900-1 906.

[2] Constantin A, Ivanov R I. On an integrable two-component Camassa-Holm shallow water system[J]. Physics Letters A, 2008, 372: 7 129-7 132.

[3] Camassa R, Holm D. An integrable shallow water equation with peaked solitons[J]. Phys Rev Lett, 1993, 71: 1 661-1 664.

[4] Constantin A, Lannes D. The hydrodynamical relevance of the Camassa-Holm and Degasperis-Procesi equations [J]. Arch Ration Mech Anal, 2009, 192: 165-186.

[5] Dullin H R, Gottwald G A, Holm D D. An integrable shallow water equation with linear and nonlinear dispersion[J]. Phys Rev Lett, 2001, 87: 4 501-4 504.

[6] Ionescu-Krus D. Variational derivation of the Camassa-Holm shallow water equation[J]. J Nonlinear Math Phys, 2007, 14: 303-312.

[7] Ivanov R I. Water waves and integrability[J]. Philos Trans R Soc Lond Ser A Math Phys Eng Sci, 2007, 365: 2 267-2 280.

[8] Johnson R S. Camassa-Holm, Korteweg-de Vries and related models for water waves[J]. J Fluid Mech, 2002, 457: 63-82.

[9] Constantin A, Strauss W A. Stability of a class of solitary waves in compressible elastic rods[J]. Phys Lett A, 2000, 270: 140-148.

[10] Dai H H. Model equations for nonlinear dispersive waves in a compressible Mooney-Rivlin rod[J]. Acta Mech, 1998, 127: 193-207.

[11] Constantin A. The Hamiltonian structure of the Camassa-Holm equation[J]. Expo Math, 1997, 15: 53-85.

[12] Fokas A, Fuchssteiner B. Symplectic structures, their Bäcklund transformation and hereditary symmetries[J]. Phys D, 1981, 4: 47-66.

[13] Constantin A. On the scattering problem for the Camassa-Holm equation[J]. Proc R Soc Lond Ser A Math Phys Eng Sci, 2001, 457: 953-970.

[14] Lakshmanan M. Tsunami and nonlinear waves[M]. Berlin: Springer, 2007.

[15] Constantin A, Johnson R S. Propagation of very long water waves, with vorticity, over variable depth, with applications to tsunamis[J]. Fluid Dynam Res, 2008, 4: 175-211.

[16] Constantin A, Escher J. Global existence and blow-up for a shallow water equation[J]. Ann Sc Norm Super Pisa Cl Sci, 1998, 26: 303-328.

[17] Constantin A, Escher J. Well-posedness, global existence and blowup phenomena for a periodic quasi-linear hyperbolic equation[J]. Comm Pure Appl Math, 1998, 51: 475-504.

[18] Danchin R. A few remarks on the Camassa-Holm equation [J]. Differential Integral Equations, 2001, 14: 953-988.

[19] Escher J, Yin Z. Initial boundary value problems of the Camassa-Holm equation [J]. Comm Partial Differential

Equations, 2008, 33: 377-395.

[20] Escher J, Yin Z. Initial boundary value problems for nonlinear dispersive wave equations[J]. J Funct Anal, 2009, 25: 479-508.

[21] Constantin A. Existence of permanent and breaking waves for a shallow water equation: a geometric approach[J]. Ann Inst Fourier (Grenoble), 2000, 50: 321-362.

[22] Constantin A, Escher J. On the blow-up rate and the blow-up of breaking waves for a shallow water equation[J]. Math Z, 2000, 233: 75-91.

[23] Escher J, Lechtenfeld O, Yin Z. Well-posedness and blow-up phenomena for the 2-component Camassa-Holm equation[J]. Discrete Contin Dyn Syst Ser A, 2007, 19: 493-513.

[24] Olver P J. Grauate texts in mathematics[M]. New York: Springer, 1993.

[25] Bluman G, Anco S. Symmetry and integration methods for differential equations [M]. New York: Springer-Verlag, 2002.

[26] Sinkala W, Leach P, Hara J O. Invariance properties of a general-pricing equation[J]. J Differential Equations, 2008, 244: 2 820-2 835.

[27] Craddock M, Lennox K. Lie group symmetries as integral transforms of fundamental solutions [J]. J Differential Equations, 2007, 232: 652-674.

[28] Liu H, Li J, Liu L. Conservation law classification and integrability of generalized nonlinear second-order equation [J]. Commun Theor Phys, 2011, 56: 987-991.

[29] Asmar N H. Partial differential equations with fourier series and boundary value problems[M]. 2nd ed. Beijing: China Machine Press, 2005.

[30] Rudin W. Principles of mathematical analysis[M]. Beijing: China Machine Press, 2004.