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On a two-parameter Hilbert-type integral operator and its applications*

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Abstract In this work, by introducing two parameters λ_1 and λ_2 and using the method of weight function and the technique of functional analysis, a two-parameter Hilbert-type integral operator is defined and the norm of the operator is given. As applications, a few improved results and some new Hilbert-type integral inequalities with the particular kernels are obtained.

Key words two-parameter Hilbert-type integral operator; norm; weight function; the best constant factor; Hilbert-type integral inequality

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一个双参数 Hilbert 型积分算子及其应用

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摘 要 引入两个参数 λ_1 和 λ_2，利用权函数方法和泛函分析技巧，定义一个双参数 Hilbert 型积分算子，并给出算子的范 数. 作为应用，获得一些改进的结果，并得到一些具有特殊核的新的 Hilbert 型积分不等式.

关键词 双参数的 Hilbert 型积分算子；范数；权函数；最佳常数因子；Hilbert 型积分不等式

If $p > 1, \frac{1}{p} + \frac{1}{q} = 1, f, g \geqslant 0, f \in L^p(0, \infty)$, $g \in L^q(0, \infty)$, $\|f\|_p = \{\int_0^\infty f^p(x)\mathrm{d}x\}^{\frac{1}{p}} > 0$, $\|g\|_q > 0$, then we have the following famous Hardy-Hilbert integral inequality and its equivalent form[1]:

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}\mathrm{d}x\mathrm{d}y < \frac{\pi}{\sin\left(\frac{\pi}{p}\right)} \|f\|_p \|g\|_q, \tag{1}$$

$$\left\{\int_0^\infty \left[\int_0^\infty \frac{f(x)}{x+y}\mathrm{d}x\right]^p \mathrm{d}y\right\}^{\frac{1}{p}} < \frac{\pi}{\sin\left(\frac{\pi}{p}\right)} \|f\|_p, \tag{2}$$

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where the constant factor $\pi/\sin(\pi/p)$ is the best possible. Inequalities (1) and (2) are important in analysis and its applications[1-2]. Define the Hardy-Hilbert's integral operator $T: L^p(0,\infty) \to L^p(0,\infty)$ as follows. For $f \in L^p(0,\infty)$, corresponding to the only

$$T(f)(y):=\int_0^\infty \frac{f(x)}{x+y}\mathrm{d}x, y \in (0,\infty), \quad (3)$$

by (2), we have $\|Tf\|_p < \pi/\sin(\pi/p)\|f\|_p$ and $\|T\| \leqslant \pi/\sin(\pi/p)$. Since the constant factor in (2) is the best possible, we find that $\|T\| = \pi/\sin(\pi/p)$[3].

In recent years, the relevant operators were defined by using bilinear function $k(x,y)(\geqslant 0)$ to replace $\frac{1}{x+y}$ in (3) by Yang et al. and they obtained two equivalent inequalities similar to (1) and (2)[4-9]. In this work, we use the nonnegative measurable function $h(x_1^\lambda y_2^\lambda)$ instead of the kernel $\frac{1}{x+y}$, and a Hilbert-type integral operator with the two-parameter kernels is defined. By using the method of weight function and the techniques of real analysis and functional analysis, the norm of the integral operator is found and two equivalent inequalities related to the norm of the operator are given. As applications, some new inequalities with the particular kernels are obtained.

1 Definitions and lemmas

We need the following special functions[10]:

Beta-function

$$B(u,v) = \int_0^\infty \frac{t^{u-1}\mathrm{d}t}{(1+t)^{u+v}} = \int_0^1 (1-t)^{u-1}t^{v-1}\mathrm{d}t (u,v>0), \quad (4)$$

Γ-function

$$\Gamma(z) = \int_0^\infty \mathrm{e}^{-u}u^{z-1}\mathrm{d}u (z>0), \quad (5)$$

Riemann's *zeta*-function

$$\zeta(x) = \sum_{k=1}^\infty \frac{1}{k^x}(x>1), \quad (6)$$

and the extended ζ-function

$$\zeta(s,a) = \sum_{k=0}^\infty \frac{1}{(k+a)^s}, \quad (7)$$

where $\mathrm{Re}(s)>1$, a is not equal to zero or negative integer. Obviously, $\zeta(s,1)=\zeta(s)$.

If a is not equal to zero or negative integer, $0<s\leqslant 1$, we define the following value of the convergent series as

$$S(s,a) = \sum_{k=0}^\infty \frac{(-1)^k}{(k+a)^s}. \quad (8)$$

Lemma 1.1 If $s>0$, a is not equal to zero or negative integer, we have the summation formula as

$$C(s,a) = \sum_{k=0}^\infty \frac{(-1)^k}{(k+a)^s} = \begin{cases} S(s,a), & 0<s\leqslant 1 \\ \frac{1}{2^s}\left[\zeta\left(s,\frac{a}{2}\right)-\zeta\left(s,\frac{a+1}{2}\right)\right], & s>1. \end{cases} \quad (9)$$

Proof When $0<s\leqslant 1$, clearly, $\sum_{k=0}^\infty \frac{(-1)^k}{(k+a)^s} = S(s,a)$, and when $s>1$, we have

$$\sum_{k=0}^\infty \frac{(-1)^k}{(k+a)^s} = \sum_{k=0}^\infty \frac{1}{(2k+a)^s} - \sum_{k=0}^\infty \frac{1}{(2k+1+a)^s} = \frac{1}{2^s}\left[\sum_{k=0}^\infty \frac{1}{\left(k+\frac{a}{2}\right)^s} - \sum_{k=0}^\infty \frac{1}{\left(k+\frac{a+1}{2}\right)^s}\right] = \frac{1}{2^s}\left[\zeta\left(s,\frac{a}{2}\right)-\zeta\left(s,\frac{a+1}{2}\right)\right].$$

If $\lambda_1, \lambda_2 > 0$, writing down $\alpha = \frac{\lambda_1+\lambda_2}{\lambda_1\lambda_2}$, $h(u)$ is a nonnegative measurable function in $(0,\infty)$, we define k_{λ_1,λ_2} as

$$k_{\lambda_1,\lambda_2} := \int_0^\infty h(u)u^{\alpha-1}\mathrm{d}u, \quad (10)$$

assuming that $k_{\lambda_1,\lambda_2}(\geqslant 0)$ is a limited number. Setting $u = x^{\lambda_1}y^{\lambda_2}$, we have

$$\omega_{\lambda_1,\lambda_2}(x) := \lambda_2 x^{\frac{\lambda_1}{\lambda_2}+1}\int_0^\infty h(x^{\lambda_1}y^{\lambda_2})y^{\frac{\lambda_2}{\lambda_1}}\mathrm{d}y = k_{\lambda_1,\lambda_2}, (x\in(0,\infty)), \quad (11)$$

$$\omega_{\lambda_1,\lambda_2}(y) := \lambda_1 y^{\frac{\lambda_1}{\lambda_2}+1}\int_0^\infty h(x^{\lambda_1}y^{\lambda_2})x^{\frac{\lambda_1}{\lambda_2}}\mathrm{d}x = k_{\lambda_1,\lambda_2}, (y\in(0,\infty)), \quad (12)$$

Lemma 1.2 Suppose that $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $\lambda_1, \lambda_2>0$, $h(u)$ is a nonnegative measurable function in $(0,\infty)$, k_{λ_1,λ_2} (see (10)) is a nonnegative and limited number, $f,\ g \geqslant 0$, satisfying $\int_0^\infty x^{-\frac{p\lambda_1}{\lambda_2}-1} f^p(x)\,\mathrm{d}x < \infty$, $\int_0^\infty y^{-\frac{q\lambda_2}{\lambda_1}-1} g^q(y)\,\mathrm{d}y < \infty$, then we have the following equivalent inequalities:

$$I_{\lambda_1,\lambda_2} := \int_0^\infty y^{\frac{\frac{q\lambda_2}{\lambda_1}+1}{q-1}} \left\{\int_0^\infty h(x^{\lambda_1}y^{\lambda_2}) f(x)\,\mathrm{d}x\right\}^p \mathrm{d}y \leqslant \left[\frac{k_{\lambda_1,\lambda_2}}{\lambda_1^{\frac{1}{q}}\lambda_2^{\frac{1}{p}}}\right]^p \int_0^\infty x^{-\frac{p\lambda_1}{\lambda_2}-1} f^p(x)\,\mathrm{d}x, \tag{13}$$

$$J_{\lambda_1,\lambda_2} := \int_0^\infty \int_0^\infty h(x^{\lambda_1}y^{\lambda_2}) f(x) g(y)\,\mathrm{d}x\mathrm{d}y \leqslant \frac{k_{\lambda_1,\lambda_2}}{\lambda_1^{\frac{1}{q}}\lambda_2^{\frac{1}{p}}} \left\{\int_0^\infty x^{-\frac{p\lambda_1}{\lambda_2}-1} f^p(x)\,\mathrm{d}x\right\}^{\frac{1}{p}} \times \left\{\int_0^\infty y^{-\frac{q\lambda_2}{\lambda_1}-1} g^q(y)\,\mathrm{d}y\right\}^{\frac{1}{q}}. \tag{14}$$

Proof　By the weighted Holder's inequality[11] and (12), we find

$$\left\{\int_0^\infty h(x^{\lambda_1}y^{\lambda_2}) f(x)\,\mathrm{d}x\right\}^p = \left\{\int_0^\infty h(x^{\lambda_1}y^{\lambda_2}) \left[\frac{y^{\frac{\lambda_2}{p\lambda_1}}}{x^{\frac{\lambda_1}{q\lambda_2}}} f(x)\right]\left[\frac{x^{\frac{\lambda_1}{q\lambda_2}}}{y^{\frac{\lambda_2}{p\lambda_1}}}\right]\mathrm{d}x\right\}^p$$

$$\leqslant \int_0^\infty h(x^{\lambda_1}y^{\lambda_2}) \frac{y^{\frac{\lambda_2}{\lambda_1}}}{x^{\frac{(p-1)\lambda_1}{\lambda_2}}} f^p(x)\,\mathrm{d}x \times \left\{\int_0^\infty h(x^{\lambda_1}y^{\lambda_2}) \frac{x^{\frac{\lambda_1}{\lambda_2}}}{y^{\frac{(q-1)\lambda_2}{\lambda_1}}}\mathrm{d}x\right\}^{p-1}$$

$$= \frac{k_{\lambda_1,\lambda_2}^{p-1}}{\lambda_1^{p-1}} y^{(1-p)\left(\frac{q\lambda_2}{\lambda_1}+1\right)} \int_0^\infty h(x^{\lambda_1}y^{\lambda_2}) \frac{y^{\frac{\lambda_2}{\lambda_1}}}{x^{\frac{(p-1)\lambda_1}{\lambda_2}}} f^p(x)\,\mathrm{d}x.$$

By Fubini's theorem[12] and (11), we have

$$I_{\lambda_1,\lambda_2} \leqslant \frac{k_{\lambda_1,\lambda_2}^{p-1}}{\lambda_1^{p-1}} \int_0^\infty \int_0^\infty h(x^{\lambda_1}y^{\lambda_2}) \frac{y^{\frac{\lambda_2}{\lambda_1}}}{x^{\frac{(p-1)\lambda_1}{\lambda_2}}} f^p(x)\,\mathrm{d}x\mathrm{d}y$$

$$= \frac{k_{\lambda_1,\lambda_2}^{p-1}}{\lambda_1^{p-1}} \int_0^\infty \left[\int_0^\infty h(x^{\lambda_1}y^{\lambda_2}) \frac{y^{\frac{\lambda_2}{\lambda_1}}}{x^{\frac{(p-1)\lambda_1}{\lambda_2}}}\mathrm{d}y\right] f^p(x)\,\mathrm{d}x$$

$$= \frac{k_{\lambda_1,\lambda_2}^{p-1}}{\lambda_1^{p-1}\lambda_2} \int_0^\infty \omega_{\lambda_1,\lambda_2}(y) x^{-\frac{p\lambda_1}{\lambda_2}-1} f^p(x)\,\mathrm{d}x$$

$$= \left[\frac{k_{\lambda_1,\lambda_2}}{\lambda_1^{\frac{1}{q}}\lambda_2^{\frac{1}{p}}}\right]^p \int_0^\infty x^{-\frac{p\lambda_1}{\lambda_2}-1} f^p(x)\,\mathrm{d}x.$$

By Hölder's inequality, we have

$$J_{\lambda_1,\lambda_2} = \int_0^\infty \left\{y^{\frac{\frac{q\lambda_2}{\lambda_1}+1}{p(q-1)}} \int_0^\infty h(x^{\lambda_1}y^{\lambda_2}) f(x)\,\mathrm{d}x\right\} \times \left\{y^{\frac{-\frac{q\lambda_2}{\lambda_1}-1}{p(q-1)}} g(y)\right\}\mathrm{d}y \leqslant I_{\lambda_1,\lambda_2}^{\frac{1}{p}} \left\{\int_0^\infty y^{-\frac{q\lambda_2}{\lambda_1}-1} g^q(y)\,\mathrm{d}y\right\}^{\frac{1}{q}}. \tag{15}$$

By (15) and (13), we obtain (14).

On the contrary, if (14) is true, for $y > 0$, setting the function as

$$g(y) := y^{\frac{\frac{q\lambda_2}{\lambda_1}+1}{q-1}} \left[\int_0^\infty h(x^{\lambda_1}y^{\lambda_2}) f(x)\,\mathrm{d}x\right]^{p-1},$$

by (14) we obtain

$$0 \leqslant \int_0^\infty y^{-\frac{q\lambda_2}{\lambda_1}-1} g^q(y)\,\mathrm{d}y = \int_0^\infty y^{\frac{\frac{q\lambda_2}{\lambda_1}+1}{q-1}} \left\{\int_0^\infty h(x^{\lambda_1}y^{\lambda_2}) f(x)\,\mathrm{d}x\right\}^p \mathrm{d}y = I_{\lambda_1,\lambda_2} = J_{\lambda_1,\lambda_2}$$

$$\leqslant \frac{k_{\lambda_1,\lambda_2}}{\lambda_1^{\frac{1}{q}}\lambda_2^{\frac{1}{p}}} \left\{\int_0^\infty x^{-\frac{p\lambda_1}{\lambda_2}-1} f^p(x)\,\mathrm{d}x\right\}^{\frac{1}{p}} \times \left\{\int_0^\infty y^{-\frac{q\lambda_2}{\lambda_1}-1} g^q(y)\,\mathrm{d}y\right\}^{\frac{1}{q}}. \tag{16}$$

By (15), we have that $I_{\lambda_1,\lambda_2} < \infty$. If $I_{\lambda_1,\lambda_2} = 0$, (13) is tenable naturally. If $0 < I_{\lambda_1,\lambda_2} < \infty$, by (16) we have

$$\left\{\int_0^\infty y^{-\frac{q\lambda_2}{\lambda_1}-1} g^q(y)\,\mathrm{d}y\right\}^{\frac{1}{p}} = I_{\lambda_1,\lambda_2}^{\frac{1}{p}} \leqslant \frac{k_{\lambda_1,\lambda_2}}{\lambda_1^{\frac{1}{q}}\lambda_2^{\frac{1}{p}}} \left\{\int_0^\infty x^{-\frac{p\lambda_1}{\lambda_2}-1} f^p(x)\,\mathrm{d}x\right\}^{\frac{1}{p}},$$

namely,

$$\int_0^\infty y^{\frac{\frac{q\lambda_2}{\lambda_1}+1}{q-1}} \left\{\int_0^\infty h(x^{\lambda_1}y^{\lambda_2}) f(x)\,\mathrm{d}x\right\}^p \mathrm{d}y \leqslant \left[\frac{k_{\lambda_1,\lambda_2}}{\lambda_1^{\frac{1}{q}}\lambda_2^{\frac{1}{p}}}\right]^p \int_0^\infty x^{-\frac{p\lambda_1}{\lambda_2}-1} f^p(x)\,\mathrm{d}x.$$

So (13) and (14) are equivalent. The lemma is proved. □

Lemma 1.3　If $p > 1, \frac{1}{p} + \frac{1}{q} = 1, \lambda_1, \lambda_2 > 0$, writing down $\alpha = \frac{\lambda_1 + \lambda_2}{\lambda_1\lambda_2}$, $h(u)$ is a nonnegative measurable function in $(0, \infty)$, satisfying that k_{λ_1,λ_2} (see (12)) is a nonnegative and finite number. If there is $\delta > 0$, such that $k_{\lambda_1,\lambda_2}, \delta = \int_0^\infty h(u) u^{\alpha+\delta-1}$ du is still a nonnegative and limited number. For 0

$< \varepsilon < \min\{p\delta, q\delta\}$, and ε is small enough, we define the following real function

$$\tilde{f}(x) := \begin{cases} 0, & x \in (0,1) \\ x^{\frac{\frac{p\lambda_1}{\lambda_2}-\lambda_1\varepsilon}{p}}, & x \in [1,\infty) \end{cases},$$

$$\tilde{g}(y) := \begin{cases} 0, & y \in (1,\infty) \\ y^{\frac{\frac{q\lambda_2}{\lambda_1}+\lambda_2\varepsilon}{q}}, & y \in (0,1] \end{cases}.$$

Then we have

$$\tilde{J}_{\lambda_1,\lambda_2}\varepsilon = \left[\int_0^\infty x^{-\frac{p\lambda_1}{\lambda_2}-1}\tilde{f}^p(x)\,dx\right]^{\frac{1}{p}} \times \left[\int_0^\infty y^{-\frac{q\lambda_2}{\lambda_1}-1}\tilde{g}^q(y)\,dy\right]^{\frac{1}{q}}\varepsilon = \frac{1}{\lambda_1^{\frac{1}{p}}\lambda_2^{\frac{1}{q}}}, \tag{17}$$

and

$$\tilde{I}_{\lambda_1,\lambda_2}\varepsilon = \varepsilon\int_0^\infty\int_0^\infty h(x^{\lambda_1}y^{\lambda_2})\tilde{f}(x)\tilde{g}(y)\,dx\,dy = \frac{k_{\lambda_1,\lambda_2}}{\lambda_1\lambda_2} + o(1)\,(\varepsilon \to 0^+). \tag{18}$$

Proof We easily get

$$\tilde{J}\varepsilon = \left[\int_0^\infty x^{-\frac{p\lambda_1}{\lambda_2}-1}\tilde{f}^p(x)\,dx\right]^{\frac{1}{p}}\left[\int_0^\infty y^{-\frac{q\lambda_2}{\lambda_1}-1}\tilde{g}^q(y)\,dy\right]^{\frac{1}{q}}\varepsilon$$
$$= \left[\int_1^\infty x^{-1-\lambda_1\varepsilon}\,dx\right]^{\frac{1}{p}}\left[\int_0^1 y^{-1+\lambda_2\varepsilon}\,dy\right]^{\frac{1}{q}}\varepsilon$$
$$= \frac{1}{\lambda_1^{\frac{1}{p}}\lambda_2^{\frac{1}{q}}}.$$

Setting $u = x^{\lambda_1}y^{\lambda_2}$, by Fubini's theorem, we have

$$\tilde{I}\varepsilon = \varepsilon\int_0^\infty\int_0^\infty h(x^{\lambda_1}y^{\lambda_2})\tilde{f}(x)\tilde{g}(y)\,dx\,dy$$
$$= \varepsilon\int_1^\infty x^{\frac{\frac{p\lambda_1}{\lambda_2}-\lambda_1\varepsilon}{p}}\,dx\left[\int_0^1 h(x^{\lambda_1}y^{\lambda_2})\,y^{\frac{\frac{q\lambda_2}{\lambda_1}+\lambda_2\varepsilon}{q}}\,dy\right]$$
$$= \frac{\varepsilon}{\lambda_2}\int_1^\infty x^{-1-\lambda_1\varepsilon}\,dx\left[\int_0^{x^{\lambda_1}} h(u)\,u^{\alpha+\frac{\varepsilon}{q}-1}\,du\right]$$
$$= \frac{\varepsilon}{\lambda_2}\int_1^\infty x^{-1-\lambda_1\varepsilon}\,dx\left[\int_0^1 h(u)\,u^{\alpha+\frac{\varepsilon}{q}-1}\,du + \int_1^{x^{\lambda_1}} h(u)\,u^{\alpha+\frac{\varepsilon}{q}-1}\,du\right]$$
$$= \frac{1}{\lambda_1\lambda_2}\int_0^1 h(u)\,u^{\alpha+\frac{\varepsilon}{q}-1}\,du + \frac{\varepsilon}{\lambda_2}\int_1^\infty x^{-1-\lambda_1\varepsilon}\,dx\left[\int_1^{x^{\lambda_1}} h(u)\,u^{\alpha+\frac{\varepsilon}{q}-1}\,du\right]$$
$$= \frac{1}{\lambda_1\lambda_2}\int_0^1 h(u)\,u^{\alpha+\frac{\varepsilon}{q}-1}\,du + \frac{\varepsilon}{\lambda_2}\int_1^\infty\left(\int_u^\infty x^{-1-\lambda_1\varepsilon}\,dx\right)h(u)\,u^{\alpha+\frac{\varepsilon}{q}-1}\,du$$
$$= \frac{1}{\lambda_1\lambda_2}\int_0^1 h(u)\,u^{\alpha+\frac{\varepsilon}{q}-1}\,du + \frac{1}{\lambda_1\lambda_2}\int_1^\infty h(u)\,u^{\alpha-\frac{\varepsilon}{p}-1}\,du. \tag{19}$$

Since $h(u)u^{\alpha+\frac{\varepsilon}{q}-1} \leqslant h(u)u^{\alpha-1}, u \in (0,1)$, $h(u)u^{\alpha-\frac{\varepsilon}{p}-1} \leqslant h(u)u^{\alpha+\delta-1}, u \in (1,\infty)$, k_{λ_1,λ_2} and $k_{\lambda_1,\lambda_2,\delta}$ are limited numbers, by Lebesgue's dominated convergence theorem[12], when $\varepsilon \to 0^+$, we obtain

$$\int_0^1 h(u)\,u^{\alpha+\frac{\varepsilon}{q}-1}\,du = \int_0^1 h(u)\,u^{\alpha-1}\,du + o_1(1). \tag{20}$$

$$\int_1^\infty h(u)\,u^{\alpha-\frac{\varepsilon}{p}-1}\,du = \int_1^\infty h(u)\,u^{\alpha-1}\,du + o_2(1). \tag{21}$$

Putting (20) and (21) into (19), we get (18). □

2 Main results and applications

If $\theta(x)(>0)$ is a measurable function, $\rho \geqslant 1$, the function space is set as

$$L_\theta^\rho(0,\infty) := \left\{h \geqslant 0;\ \|h\|_{\rho,\theta} := \left\{\int_0^\infty \theta(x)h^\rho(x)\,dx\right\}^{\frac{1}{\rho}} < \infty\right\}.$$

If $p > 1, \frac{1}{p}+\frac{1}{q} = 1, \lambda_1, \lambda_2 > 0$, writing down $\alpha = \frac{\lambda_1+\lambda_2}{\lambda_1\lambda_2}$, $h(u)$ is a nonnegative measurable function in $(0,\infty)$, satisfying that k_{λ_1,λ_2} (see (10)) is a nonnegative and finite number, setting $\varphi(x) := x^{-\frac{p\lambda_1}{\lambda_2}-1}, \psi(y) := y^{-\frac{q\lambda_2}{\lambda_1}-1}, x, y \in (0,\infty)$, and $\psi^{1-p}(y) = y^{\frac{\frac{q\lambda_2}{\lambda_1}+1}{q-1}}$, we define an operator $T: L_\varphi^p(0,\infty) \to L_{\psi^{1-p}}^p(0,\infty)$, for $f \in L_\varphi^p(0,\infty)$

$$(Tf)(y) := \int_0^\infty h(x^{\lambda_1}y^{\lambda_2})f(x)\,dx,\ (y \in (0,\infty)). \tag{22}$$

In view of (13), it follows $Tf \in L_{\psi^{1-p}}^p$. For $g \in L_\psi^q(0,\infty)$, we define the formal inner of Tf and g

$$(Tf, g) := \int_0^\infty\int_0^\infty h(x^{\lambda_1}y^{\lambda_2})f(x)g(y)\,dx\,dy. \tag{23}$$

Hence the equivalent inequalities (13) and (14) may be rewritten in the following abstract forms

$$\| Tf \|_{p,\psi^{1-p}}^{p} \leqslant \left[\frac{k_{\lambda_1,\lambda_2}}{\lambda_1^{\frac{1}{q}} \lambda_2^{\frac{1}{p}}} \right]^p \| f \|_{p,\varphi}^{p}, \qquad (24)$$

$$(Tf, g) \leqslant \frac{k_{\lambda_1,\lambda_2}}{\lambda_1^{\frac{1}{q}} \lambda_2^{\frac{1}{p}}} \| f \|_{p,\varphi} \| g \|_{q,\psi}. \qquad (25)$$

T is obviously bounded[13], and $\| T \| \leqslant \frac{k_{\lambda_1,\lambda_2}}{\lambda_1^{\frac{1}{q}} \lambda_2^{\frac{1}{p}}}$. We call that T is two-parameters Hilbert-type integral operator.

Theorem 2.1 If $p > 1, \frac{1}{p} + \frac{1}{q} = 1, \lambda_1, \lambda_2 > 0$, $h(u)$ is a nonnegative measurable function in $(0, \infty)$, k_{λ_1,λ_2} (see (10)) is a nonnegative and limited number, $\varphi(x) = x^{-\frac{p\lambda_1}{\lambda_2}-1}, \psi(y) = y^{-\frac{q\lambda_2}{\lambda_1}-1}, f \in L_{\varphi}^{p}(0, \infty), g \in L_{\psi}^{q}(0, \infty)$, $\| f \|_{p,\varphi}, \| g \|_{q,\psi} > 0$, then we have $\| T \| = \frac{k_{\lambda_1,\lambda_2}}{\lambda_1^{\frac{1}{q}} \lambda_2^{\frac{1}{p}}}$, and the constant factors in (24) and (25) are the best possible.

Proof Assuming that the constant factor $\frac{k_{\lambda_1,\lambda_2}}{\lambda_1^{\frac{1}{q}} \lambda_2^{\frac{1}{p}}}$ in (25) is not the best possible, then there exists a positive $k < \frac{k_{\lambda_1,\lambda_2}}{\lambda_1^{\frac{1}{q}} \lambda_2^{\frac{1}{p}}}$, such that inequality (25) is still valid if we replace $\frac{k_{\lambda_1,\lambda_2}}{\lambda_1^{\frac{1}{q}} \lambda_2^{\frac{1}{p}}}$ by k. Specially, putting $\bar{f}(x)$ and $\bar{g}(y)$ in Lemma 1.3 instead of $f(x)$ and $g(y)$, then by (17) and (18) we have

$$\frac{k_{\lambda_1,\lambda_2}}{\lambda_1 \lambda_2} + o(1) < \frac{k}{\lambda_1^{\frac{1}{p}} \lambda_2^{\frac{1}{q}}}.$$

Letting $\varepsilon \to 0^+$, we get $k \geqslant \frac{k_{\lambda_1,\lambda_2}}{\lambda_1^{\frac{1}{q}} \lambda_2^{\frac{1}{p}}}$, which contradicts the fact that $k < \frac{k_{\lambda_1,\lambda_2}}{\lambda_1^{\frac{1}{q}} \lambda_2^{\frac{1}{p}}}$. So the constant factor $\frac{k_{\lambda_1,\lambda_2}}{\lambda_1^{\frac{1}{q}} \lambda_2^{\frac{1}{p}}}$ in (25) is the best possible one. If the constant factor $\left[\frac{k_{\lambda_1,\lambda_2}}{\lambda_1^{\frac{1}{q}} \lambda_2^{\frac{1}{p}}} \right]^p$ in (24) is not the best possible, then by (15) we get a contradictory conclusion that the constant factor $\frac{k_{\lambda_1,\lambda_2}}{\lambda_1^{\frac{1}{q}} \lambda_2^{\frac{1}{p}}}$ in (25) is not the best possible. So, $\| T \| = \frac{k_{\lambda_1,\lambda_2}}{\lambda_1^{\frac{1}{q}} \lambda_2^{\frac{1}{p}}}$. □

Theorem 2.2 As Theorem 2.1, inequalities (24) and (25) keep the strict forms, namely,

$$\| Tf \|_{p,\psi^{1-p}}^{p} = \int_0^\infty y^{\frac{\frac{q\lambda_2}{\lambda_1}+1}{q-1}} \left\{ \int_0^\infty h(x^{\lambda_1} y^{\lambda_2}) f(x) \mathrm{d}x \right\}^p \mathrm{d}y$$
$$< \left[\frac{k_{\lambda_1,\lambda_2}}{\lambda_1^{\frac{1}{q}} \lambda_2^{\frac{1}{p}}} \right]^p \| f \|_{p,\varphi}^{p}, \qquad (26)$$

$$(Tf, g) = \int_0^\infty \int_0^\infty h(x^{\lambda_1} y^{\lambda_2}) f(x) g(y) \mathrm{d}x \mathrm{d}y$$
$$< \frac{k_{\lambda_1,\lambda_2}}{\lambda_1^{\frac{1}{q}} \lambda_2^{\frac{1}{p}}} \| f \|_{p,\varphi} \| g \|_{q,\psi}, \qquad (27)$$

where the constant factors $\left[\frac{k_{\lambda_1,\lambda_2}}{\lambda_1^{\frac{1}{q}} \lambda_2^{\frac{1}{p}}} \right]^p$ and $\frac{k_{\lambda_1,\lambda_2}}{\lambda_1^{\frac{1}{q}} \lambda_2^{\frac{1}{p}}}$ are the best possible ones.

Proof If inequality (25) keeps the form of an equality, by Lemma1.2 there exist two constants A and B such that they are not all zeroes[11], and they satisfy

$$A \frac{y^{\frac{\lambda_2}{\lambda_1}}}{x^{\frac{p\lambda_1}{q\lambda_2}}} f^p(x) = B \frac{x^{\frac{\lambda_1}{\lambda_2}}}{y^{\frac{q\lambda_2}{p\lambda_1}}} g^q(y) \text{ a. e. in} (0, \infty) \times (0, \infty).$$

It follows that $A x^{-\frac{p\lambda_1}{\lambda_2}} f^p(x) = B y^{-\frac{q\lambda_2}{\lambda_1}} g^q(y)$ a. e. in $(0, \infty) \times (0, \infty)$. Assuming that $A \neq 0$, there exists $y > 0$, such that $x^{-\frac{p\lambda_1}{\lambda_2}-1} f^p(x) = \left[B y^{-\frac{q\lambda_2}{\lambda_1}} g^q(y) \right] \frac{1}{Ax}$ a. e. in $(0, \infty)$, which contradicts the fact that $0 < \| f \|_{p,\varphi} < \infty$. Then inequality (25) keeps the strict form. So (27) is true. Since inequalities (24) and (25) are equivalent, inequality (24) keeps the strict form too. So (26) is true. By Theorem 2.1, the constant factors $\left[\frac{k_{\lambda_1,\lambda_2}}{\lambda_1^{\frac{1}{q}} \lambda_2^{\frac{1}{p}}} \right]^p$ and $\frac{k_{\lambda_1,\lambda_2}}{\lambda_1^{\frac{1}{q}} \lambda_2^{\frac{1}{p}}}$ are the best possible ones. □

Assuming that $p > 1, \frac{1}{p} + \frac{1}{q} = 1, \lambda_1, \lambda_2 > 0$, $\alpha = \frac{\lambda_1 + \lambda_2}{\lambda_1 \lambda_2}, \varphi(x) = x^{-\frac{p\lambda_1}{\lambda_2}-1}, \psi(y) = y^{-\frac{q\lambda_2}{\lambda_1}-1}$,

$f \in L_\varphi^p(0,\infty)$, $g \in L_\psi^q(0,\infty)$, $\|f\|_{p,\varphi}$, $\|g\|_{q,\psi} > 0$, we obtain some useful inequalities by selecting some special kernels in (24) and (25).

1) If $h(u) = \frac{1}{(1+u)^\beta}$, $\beta > \alpha$, such that $k_{\lambda_1,\lambda_2} = \int_0^\infty \frac{1}{(1+u)^\beta} u^{\alpha-1} \mathrm{d}u = B(\beta-\alpha,\alpha)$, by Theorem 2.2 we have the following equivalent inequalities with the best constant factors

$$\int_0^\infty y^{\frac{\frac{q\lambda_2}{\lambda_1}+1}{q-1}} \left[\int_0^\infty \frac{f(x)\mathrm{d}x}{(1+x^{\lambda_1}y^{\lambda_2})^\beta}\right]^p \mathrm{d}y < \left[\frac{B(\beta-\alpha,\alpha)}{\lambda_1^{\frac{1}{q}}\lambda_2^{\frac{1}{p}}}\right]^p \|f\|_{p,\varphi}^p, \tag{28}$$

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)\mathrm{d}x\mathrm{d}y}{(1+x^{\lambda_1}y^{\lambda_2})^\beta} < \frac{B(\beta-\alpha,\alpha)}{\lambda_1^{\frac{1}{q}}\lambda_2^{\frac{1}{p}}} \|f\|_{p,\varphi}\|g\|_{q,\psi}. \tag{29}$$

2) If $h(u) = \frac{1}{(\max\{1,u\})^\beta}$, $\beta > \alpha$, such that $k_{\lambda_1,\lambda_2} = \int_0^\infty \frac{1}{(\max\{1,u\})^\beta} u^{\alpha-1}\mathrm{d}u = \frac{1}{\alpha(\beta-\alpha)}$, by Theorem 2.2 we have the following equivalent inequalities with the best constant factors as

$$\int_0^\infty y^{\frac{\frac{q\lambda_2}{\lambda_1}+1}{q-1}} \left[\int_0^\infty \frac{f(x)\mathrm{d}x}{(\max\{1,x^{\lambda_1}y^{\lambda_2}\})^\beta}\right]^p \mathrm{d}y < \left[\frac{1}{\lambda_1^{\frac{1}{q}}\lambda_2^{\frac{1}{p}}a(\beta-a)}\right]^p \|f\|_{p,\varphi}^p, \tag{30}$$

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)\mathrm{d}x\mathrm{d}y}{(\max\{1,x^{\lambda_1}y^{\lambda_2}\})^\beta} < \frac{1}{\lambda_1^{\frac{1}{q}}\lambda_2^{\frac{1}{p}}\alpha(\beta-\alpha)} \|f\|_{p,\varphi}\|g\|_{q,\psi}. \tag{31}$$

3) If $h(u) = \operatorname{sech} u$, by Lemma 1.1 we get that $k_{\lambda_1,\lambda_2} = \int_0^\infty (\operatorname{sech} u) u^{\alpha-1}\mathrm{d}u = 2\int_0^\infty \frac{\mathrm{e}^{-u}}{1+\mathrm{e}^{-2u}} u^{\alpha-1}\mathrm{d}u = 2\sum_{k=0}^\infty (-1)^k \int_0^\infty \mathrm{e}^{-(2k+1)u} u^{\alpha-1}\mathrm{d}u = 2\Gamma(\alpha)\sum_{k=0}^\infty \frac{(-1)^k}{(2k+1)^\alpha} = \frac{1}{2^{\alpha-1}}\Gamma(\alpha)C\left(\alpha,\frac{1}{2}\right)$, and by Theorem 2.2 we have the following equivalent inequalities with the best constant factor

$$\int_0^\infty y^{\frac{\frac{q\lambda_2}{\lambda_1}+1}{q-1}} \left[\int_0^\infty (\operatorname{sech} x^{\lambda_1}y^{\lambda_2}) f(x)\mathrm{d}x\right]^p \mathrm{d}y < \left[\frac{\Gamma(\alpha)C\left(\alpha,\frac{1}{2}\right)}{2^{\alpha-1}\lambda_1^{\frac{1}{q}}\lambda_2^{\frac{1}{p}}}\right]^p \|f\|_{p,\varphi}^p, \tag{32}$$

$$\int_0^\infty\int_0^\infty (\operatorname{sech} x^{\lambda_1}y^{\lambda_2}) f(x)g(y)\mathrm{d}x\mathrm{d}y < \frac{\Gamma(\alpha)C\left(\alpha,\frac{1}{2}\right)}{2^{\alpha-1}\lambda_1^{\frac{1}{q}}\lambda_2^{\frac{1}{p}}} \|f\|_{p,\varphi}\|g\|_{q,\psi}. \tag{33}$$

4) If $h(u) = \operatorname{csch} u$, $\alpha > 1$ $(\lambda_1+\lambda_2 > \lambda_1\lambda_2)$, by (7) we get that $k_{\lambda_1,\lambda_2} = \int_0^\infty (\operatorname{csch} u) u^{\alpha-1}\mathrm{d}u = 2\int_0^\infty \frac{\mathrm{e}^{-u}}{1-\mathrm{e}^{-2u}} u^{\alpha-1}\mathrm{d}u = 2\sum_{k=0}^\infty \int_0^\infty \mathrm{e}^{-(2k+1)u} u^{\alpha-1}\mathrm{d}u = 2\Gamma(\alpha)\sum_{k=0}^\infty \frac{1}{(2k+1)^\alpha} = \frac{\Gamma(\alpha)\zeta\left(\alpha,\frac{1}{2}\right)}{2^{\alpha-1}}$, and by Theorem 2.2 we have the following equivalent inequalities with the best constant factors

$$\int_0^\infty y^{\frac{\frac{q\lambda_2}{\lambda_1}+1}{q-1}} \left[\int_0^\infty (\operatorname{csch} x^{\lambda_1}y^{\lambda_2}) f(x)\mathrm{d}x\right]^p \mathrm{d}y < \left[\frac{\Gamma(\alpha)\zeta\left(\alpha,\frac{1}{2}\right)}{2^{\alpha-1}\lambda_1^{\frac{1}{q}}\lambda_2^{\frac{1}{p}}}\right]^p \|f\|_{p,\varphi}^p, \tag{34}$$

$$\int_0^\infty\int_0^\infty (\operatorname{csch} x^{\lambda_1}y^{\lambda_2}) f(x)g(y)\mathrm{d}x\mathrm{d}y < \frac{\Gamma(\alpha)\zeta\left(\alpha,\frac{1}{2}\right)}{2^{\alpha-1}\lambda_1^{\frac{1}{q}}\lambda_2^{\frac{1}{p}}} \|f\|_{p,\varphi}\|g\|_{q,\psi}. \tag{35}$$

5) If $h(u) = \ln(1+\mathrm{e}^{-u})$, by Lemma 1.1 we get that $k_{\lambda_1,\lambda_2} = \int_0^\infty \ln(1+\mathrm{e}^{-u}) u^{\alpha-1}\mathrm{d}u = \sum_{k=0}^\infty \frac{(-1)^k}{k+1}\int_0^\infty \mathrm{e}^{-(k+1)u} u^{a-1}\mathrm{d}u = \Gamma(\alpha)\sum_{k=0}^\infty \frac{(-1)^k}{(k+1)^{\alpha+1}} = \frac{\Gamma(\alpha)}{2^{\alpha+1}}\left[\zeta\left(\alpha+1,\frac{1}{2}\right) - \zeta\left(\alpha+1,\frac{3}{2}\right)\right]$, and by Theorem 2.2 we have the following equivalent inequalities with the best constant factors

$$\int_0^\infty y^{\frac{\frac{q\lambda_2}{\lambda_1}+1}{q-1}} \left\{\int_0^\infty [\ln(1+\mathrm{e}^{-x^{\lambda_1}y^{\lambda_2}})] f(x)\mathrm{d}x\right\}^p \mathrm{d}y < \left\{\frac{\Gamma(\alpha)\left[\zeta\left(\alpha+1,\frac{1}{2}\right) - \zeta\left(\alpha+1,\frac{3}{2}\right)\right]}{2^{\alpha+1}\lambda_1^{\frac{1}{q}}\lambda_2^{\frac{1}{p}}}\right\}^p \|f\|_{p,\varphi}^p, \tag{36}$$

$$\int_0^\infty\int_0^\infty [\ln(1+\mathrm{e}^{-x^{\lambda_1}y^{\lambda_2}})] f(x)g(y)\mathrm{d}x\mathrm{d}y$$

$$< \frac{\Gamma(\alpha)\left[\zeta\left(\alpha+1,\frac{1}{2}\right)-\zeta\left(\alpha+1,\frac{3}{2}\right)\right]}{2^{\alpha+1}\lambda_1^{\frac{1}{q}}\lambda_2^{\frac{1}{p}}}\|f\|_{p,\varphi}\|g\|_{q,\psi}. \tag{37}$$

6) If $h(u) = \arctan(e^{-u})$, by Lemma 1.1 we get that $k_{\lambda_1,\lambda_2} = \int_0^\infty \arctan(e^{-u})u^{\alpha-1}du = \sum_{k=0}^\infty \frac{(-1)^k}{(2k+1)^{\alpha+1}}\int_0^\infty e^{-u}u^{\alpha-1}du = \frac{\Gamma(\alpha)}{4^{\alpha+1}}\left[\zeta\left(\alpha+1,\frac{1}{4}\right)-\zeta\left(\alpha+1,\frac{3}{4}\right)\right]$, and by Theorem 2.2 we have the following equivalent inequalities with the best constant factors as

$$\int_0^\infty y^{\frac{\frac{q\lambda_2}{\lambda_1}+1}{q-1}}\left\{\int_0^\infty [\arctan(e^{-x^{\lambda_1}y^{\lambda_2}})]f(x)dx\right\}^p dy$$
$$< \left\{\frac{\Gamma(\alpha)\left[\zeta\left(\alpha+1,\frac{1}{4}\right)-\zeta\left(\alpha+1,\frac{3}{4}\right)\right]}{4^{\alpha+1}\lambda_1^{\frac{1}{q}}\lambda_2^{\frac{1}{p}}}\right\}^p \|f\|_{p,\varphi}^p, \tag{38}$$

$$\int_0^\infty\int_0^\infty [\arctan(e^{-x^{\lambda_1}y^{\lambda_2}})]f(x)g(y)dxdy$$
$$< \frac{\Gamma(\alpha)\left[\zeta\left(\alpha+1,\frac{1}{4}\right)-\zeta\left(\alpha+1,\frac{3}{4}\right)\right]}{4^{\alpha+1}\lambda_1^{\frac{1}{q}}\lambda_2^{\frac{1}{p}}}\cdot \|f\|_{p,\varphi}\|g\|_{q,\psi}. \tag{39}$$

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