Study on stability of systems with forced term[*]

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（Received 7 July 2016；Revised 8 November 2016）

Abstract　In this work，the system with forced term is considered. Two methods for adding the forced term to the differential dynamical systems for asymptotic stability are given. The first method is based on Lyapunov's direct method，and the second method is a new method based on stability analysis of Hopf bifurcations. The new method has a discriminant which is a rational expression consisting of the coefficients in the original system. Some further researches for adding forced term are also presented.

Keywords　asymptotic stability；forced term；nonlinear system；Hopf bifurcation

关于带有强迫项系统稳定性的研究

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摘　要　研究带有强迫项的系统，给出微分动力系统添加强迫项使其渐进稳定的两种方法。第一种方法建立在李雅普诺夫直接法的基础上；第二种方法则是基于 Hopf 分叉系统稳定性研究的新方法。这个新方法的判断依据是一个由原系统系数组成的有理判别式。最后，给出关于添加强迫项的一些进一步的研究结果。

关键词　渐近稳定性；强迫项；非线性动力系统；Hopf 分叉

Any actual system is always moving or working under a variety of occasional and ongoing disturbances. When the system is under disturbance，whether the system can safely keep the motion track or working condition，that is，the stability of the system is the most important consideration. The stability of a system includes the stability of the equilibrium state and the stability of any motion. The stability of a given motion can be transformed into the stability of the equilibrium point. The stability of equilibrium point is defined as Lyapunov stability，uniform stability，asymptotic stability，uniform asymptotic stability，exponential asymptotic stability，or global asymptotic stability.

The best general reference here is Ref. [1]. It is very important for a system to have good stability properties. However, some systems are not stable. When a system is not stable, our idea in this work is to add a forced term to the system, which makes the system have good stability properties, such as global asymptotic stability or asymptotic stability. We can adjust the form of the forced term to make it physically meaningful and be easily added to the actual system. This idea comes from the study of fuzzy logic control systems. Fuzzy logic controllers was proposed long ago and applied successfully in many applications[2-7]. A comprehensive work on the proof of stability of fuzzy logic control systems represents one of the challenges in fuzzy control[8-11]. This work is based on the research on the fuzzy logic control systems. It is innovative to add a physical forced term to the system for the asymptotic stability. To add the suitable forced terms, we propose two new methods, which are dependent upon the analyses of the stability. For this purpose, we introduce various methods for studying the stability of systems.

1　Systems with forced term

The structure of a system with forced term is presented in this section. Let X be the universe of discourse and consider a system with $x = 0$ an equilibrium point of the following form：

$$\dot{x} = f(x), x \in R^n, f(x) = (f_1(x), f_2(x), \cdots, f_n(x))^T.$$

(1)

If system (1) is not stable, we want to add the forced term to the system (the external effect in the actual problem) to make the system asymptotically stable and have good properties. The problem we need to study is the form of the forced term. In view of the actual physical problems, for the convenience of adding an external effect in the actual operation, we assume that the form of adding items as

$$D(x) = b(x)u(x),$$

$$b(x) = (b_1(x), b_2(x), \cdots, b_n(x))^T,$$

where $u(x)$ is a scalar function.

Then, we consider the stability of a system with forced term of the form

$$\dot{x} = f(x) + D(x) = f(x) + b(x)u(x). \quad (2)$$

In the study of an actual physical problem, for the convenience of adding external effects, the form of $b(x)$ in system (2) is given in advance. So the next work is to look for $u(x)$ which makes system (2) asymptotically stable while $b(x)$ is given in a different form.

2　Stability analysis of general systems with forced term

The stability analysis presented in this section is based on LaSalle's invariance principle cited and analyzed in Refs. [12-15]. This section is concentrated on the formulation and proof of Theorem 2.1 that ensures sufficient conditions for the stability of general autonomous system.

The Lyapunov function candidate $V: R^n \to R$, $V(x) = x^T P x$ is considered. It is positive and unbounded, and $P \in R^{n \times n}$ is a positive definite matrix. Then, because P is a positive define matrix, $P = Q^T Q$, where Q is an upper triangular matrix in which the diagonal elements are positive. Next we introduce a kind of linear transformation $y = Qx$ to system (2), and then it becomes the system,

$$\dot{y} = g(y) + c(y)v(y), \quad (3)$$

where $g(y) = Qf(Q^{-1}y)$, $c(y) = Qb(Q^{-1}y)$, and $v(y) = u(Q^{-1}y)$.

So we only need to obtain $v(y)$ in this system, and $u(x)$ can be obtained by the linear transformation $y = Qx$. Now we know $V(x) = x^T P x = y^T y$. Consider the derivatives of V with respect to time expressed in terms of (4)：

$$\begin{aligned}
\Omega &= \dot{V}(x) \\
&= 2(\dot{y}_1 y_1 + \cdots + \dot{y}_n y_n) \\
&= 2(y^T g(y) + y^T c(y)v(y)) \\
&= G(y) + C(y)v(y),
\end{aligned} \quad (4)$$

where　$G(y) = 2y^T g(y)$, $C(y) = 2y^T c(y)$.

The following sets are defined to be used in the stability analysis：

$$C^0 = \{y \in Y \mid C(y) = 0\}, C^+ = \{y \in Y \mid C(y) > 0\}, C^- = \{y \in Y \mid C(y) < 0\},$$

where Y is the universe transformed from X by $y = Qx$.

Theorem 2.1　　Suppose that $y = 0$ is an

equilibrium point of system （3）, in other words, $x = 0$ is an equilibrium point of system （2）. If the following conditions hold:

1) $G(y) \leqslant 0, \forall y \in C^0$;

2) $v(y) \leqslant -\dfrac{G(y)}{C(y)}, \forall y \in C^+; v(y) \geqslant -\dfrac{G(y)}{C(y)},$ $\forall y \in C^-$;

3) Set $\{y \in Y | \Omega = 0\}$ contains no state trajectories except the trivial one, $y(t) = 0, t \geqslant 0$.

Then system （2） is globally asymptotically stable in the sense of Lyapunov at the origin.

Proof From the definition of V it results that $V(0) = 0, V(x) > 0, \forall x \neq 0$ and $V(x) = x^{\mathrm{T}} P x \to \infty$ as $\| x \| \to \infty$. Further it will be proved that $\dot{V}$ is negative semi-definite with respect to time by employing （4）. An arbitrary initial state vector $x_0 \in X$ is considered. Then $y_0 = Q x_0$. The following three cases are possible.

Case 1: If $y_0 \in C^0$, it results that

$$\dot{V}(x_0) = G(y_0) + C(y_0) v(y_0) = G(y_0) \leqslant 0.$$

Case 2: If $y_0 \in C^+$, from the condition 2) of Theorem 2.1 it results that

$$v(y_0) \leqslant -\frac{G(y_0)}{C(y_0)}$$

$$\Rightarrow \dot{V}(x_0) = G(y_0) + C(y_0) v(y_0) \leqslant G(y_0) + C(y_0)\left(-\frac{G(y_0)}{C(y_0)}\right) = 0.$$

Therefore,

$$v(y_0) \leqslant -\frac{G(y_0)}{C(y_0)} \Rightarrow \dot{V}(x_0) \leqslant 0.$$

Case 3: If $y_0 \in C^-$, from the condition 2) of Theorem 2.1 it results that

$$v(y_0) \geqslant -\frac{G(y_0)}{C(y_0)}$$

$$\Rightarrow \dot{V}(x_0) = G(y_0) + C(y_0) v(y_0) \leqslant G(y_0) + C(y_0)\left(-\frac{G(y_0)}{C(y_0)}\right) = 0.$$

Therefore,

$$v(y_0) \geqslant -\frac{G(y_0)}{C(y_0)} \Rightarrow \dot{V}(x_0) \leqslant 0.$$

In the above three cases it is obtained that

$$\dot{V}(x) \leqslant 0, \forall x \in X.$$

In conclusion, the derivative with respect to time of

the Lyapunov function candidate, $\dot{V}$, is negative semi-definite.

Because $y = Qx$, from the condition 3) we know that set $\{x \in X | \dot{V}(x) = 0\}$ contains no state trajectories except the trivial one, $x(t) = 0, t \geqslant 0$. The condition 3) ensures the fulfilment of LaSalle's invariance principle. This justifies the fact that the equilibrium point at the origin is globally asymptotically stable. The proof is now completed.

The conditions 1) and 2) in Theorem 2.1 guarantee that the function $\dot{V}$ is negative semi-definite. The condition 3) proves that set $\{0\}$ is the largest invariance set in $\{x \in X | \dot{V}(x) = 0\}$. By LaSalle's invariance principle it has been guaranteed that system （2） is globally asymptotically stable in the sense of Lyapunov at the origin.

The stability analysis algorithm ensuring the stability of the systems with forced term is based on Theorem 2.1. It consists of five steps shown as follows.

1) Give a general upper triangular matrix in which the diagonal elements are positive,

$$Q = \begin{pmatrix} a_{11} & \cdot & \cdot & a_{1n} \\ & \cdot & & \cdot \\ & & \cdot & \cdot \\ & & & a_{nn} \end{pmatrix}_{n \times n}; \ a_{11}, \cdots, a_{nn} > 0.$$

2) Transform system （1） to system （2） by the linear transformation $y = Qx$ and determine $G(y)$, $C(y)$ and C^0.

3) Apply the undetermined coefficient method to determine the elements of Q by the condition 1) of Theorem 2.1 and then determine C^+ and C^-.

4) Determine $v(y)$ so that $v(y) \leqslant -\dfrac{G(y)}{C(y)}$ for $y \in C^+$, $v(y) \geqslant -\dfrac{G(y)}{C(y)}$ for $y \in C^-$, and $c(y) v(y) \to 0$ if $y \to 0$, and $v(y)$ should fulfill the condition 3) of Theorem 2.1.

5) Use the linear transformation $y = Qx$ to find $u(x)$ after $v(y)$ is obtained.

Note that the construction of $v(y)$ is based on personal attempt. However, the verification and complex computations can be implemented on a

computer. The specific application of this algorithm will be illustrated in section 6.

3 Stability analysis of Hopf bifurcations with forced term

A discriminant for judging the stability of Hopf bifurcation is presented in this section. The stability analysis of Hopf bifurcation with forced term is based on the discriminant. Consider a system with Hopf bifurcation,

$$\dot{x} = f(x) = Jx + F(x), \ x \in R^n, \quad (5)$$

where J is Jacobian matrix of system (5), f is analytic, $F(x) = (f_1, f_2, \cdots, f_n)^T$.

Note that $x = 0$ is an equilibrium of system (5). The function F and its first-order derivative vanish at $x = 0$. Without loss of generality, we assume that J is Jordan canonical form (otherwise we change J to Jordan canonical form by linear transformation $x = Ty$). Here, J has a pair of purely imaginary eigenvalues $\pm i\omega_c$ at the equilibrium $x = 0$. Without loss of generality, we assume $\omega_c = 1$ (otherwise one may use an additional transformation $t' = \omega_c t$ to change frequency ω_c to 1), and the Jordan canonical form of Jacobian matrix of system (5) at $x = 0$ is

$$J = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & A \end{bmatrix}, \quad A \in R^{(n-2) \times (n-2)},$$

where A is hyperbolic. For most of the physical situations, we assume the unstable manifold is empty (the eigenvalues of A only have negative real parts).

To give the following theorem for determining the stability of n-dimensional system conveniently, f_k in (5) is written as

$$f_1 = s_{11}x_1^2 + s_{12}x_2^2 + s_{13}x_1x_2 + s_{14}x_1x_3 + s_{15}x_2x_3 + s_{16}x_1^3 + f_{11},$$

$$f_2 = s_{21}x_1^2 + s_{22}x_2^2 + s_{23}x_1x_2 + s_{24}x_1x_3 + s_{25}x_2x_3 + s_{26}x_2^3 + f_{21},$$

$$f_k = s_{k1}x_1^2 + s_{k2}x_2^2 + s_{k3}x_1x_2 + f_{k1}, k = 3, 4, \cdots, n,$$
$$(6)$$

where f_{11} does not contain terms like x_1^2, x_2^2, x_1x_2, x_1x_3, x_2x_3, x_1^3, f_{21} does not contain terms like x_1^2, $x_2^2, x_1x_2, x_1x_3, x_2x_3, x_2^3$, and f_{k1} does not contain terms like x_1^2, x_2^2, x_1x_2.

Theorem 3.1 For system (5), if $\Delta < 0$, then the system will be unstable at the origin; if $\Delta > 0$, then the system will be asymptotically stable at the origin. Δ is given by the following formula,

$$\Delta = \sum_{i=1}^{n} \left(\frac{1}{2}\xi_{i2}B_{i2} - \xi_{i1}\left(B_{i0} + \frac{1}{2}B_{i1}\right)\right) +$$

$$\sum_{j=1}^{n} \left(\frac{1}{2}\tau_{j1}B_{j2} - \tau_{j2}\left(B_{j0} - \frac{1}{2}B_{j1}\right)\right) +$$

$$\frac{1}{2}(B_{22}\xi_{22} + B_{12}\tau_{11}) - \left(B_{10} + \frac{1}{2}B_{11}\right)\xi_{11} -$$

$$\left(B_{20} - \frac{1}{2}B_{21}\right)\tau_{22} - \frac{3}{4}(s_{16} + s_{26}), \quad (7)$$

where ξ_{i1} and ξ_{i2} are the coefficients before x_1x_i and x_2x_i in f_1, respectively, τ_{j1} and τ_{j2} are the coefficients before x_1x_j and x_2x_j in f_2, respectively,

$$B_{10} = \frac{s_{21} + s_{22}}{2}, B_{11} = \frac{s_{22} + 2s_{13} - s_{21}}{6},$$

$$B_{12} = \frac{s_{23} + 2s_{11} - 2s_{12}}{6},$$

$$B_{20} = -\frac{s_{11} + s_{12}}{2}, B_{21} = \frac{2s_{23} + s_{11} - s_{12}}{6},$$

$$B_{22} = \frac{2s_{21} - 2s_{22} - s_{13}}{6},$$

$$B_{a0} = \frac{1}{\alpha_1}B_{a+1,0} + \frac{s_{a1} + s_{a2}}{2\alpha_1},$$

$$B_{a1} = \frac{\alpha_1 B_{a+1,1} - 2B_{a+1,2}}{\alpha_1^2 + 4} + \frac{\alpha_1 s_{a1} - \alpha_1 s_{a2} + 2s_{a3}}{2(\alpha_1^2 + 4)},$$

$$B_{a2} = \frac{2B_{a+1,1} + \alpha_1 B_{a+1,2}}{\alpha_1^2 + 4} + \frac{2s_{a1} - 2s_{a2} - \alpha_1 s_{a3}}{2(\alpha_1^2 + 4)},$$

$$a = 3, 4, \cdots, k_1 + 2,$$

$$B_{b0} = \frac{s_{b1} + s_{b2}}{2\alpha_1}, B_{b1} = \frac{\alpha_1 s_{b1} - \alpha_1 s_{b2} + 2s_{b3}}{2(\alpha_1^2 + 4)},$$

$$B_{b2} = \frac{2s_{b1} - 2s_{b2} - \alpha_3 s_{b3}}{2(\alpha_1^2 + 4)}, b = k_1 + 3, \cdots, k_2 + 2,$$

$$B_{p0} = \frac{s_{p1} + s_{p2}}{2\alpha_p}, B_{p1} = \frac{\alpha_p s_{p1} - \alpha_p s_{p2} + 2s_{p3}}{2(\alpha_p^2 + 4)},$$

$$B_{p2} = \frac{2s_{p1} - 2s_{p2} - \alpha_3 s_{p3}}{2(\alpha_p^2 + 4)}, p = k_2 + 3, \cdots, k_3,$$

$$B_{c0} = \frac{\alpha B_{c+2,0} + \omega B_{c+3,0}}{\omega^2 + \alpha^2} + \frac{\alpha(s_{c1} + s_{c2}) + \omega(s_{c+1,1} + s_{c+1,2})}{2(\omega^2 + \alpha^2)},$$

$$B_{c1} = \frac{-4\alpha v_{c2} + (\alpha^2 + \omega^2 - 4)v_{c1}}{16\alpha^2 + (\alpha^2 + \omega^2 - 4)^2},$$

$$B_{c2} = \frac{4\alpha v_{c1} + (\alpha^2 + \omega^2 - 4)v_{c2}}{16\alpha^2 + (\alpha^2 + \omega^2 - 4)^2},$$

$$v_{c1} = 2B_{c+2,2} + \omega B_{c+3,1} + \alpha B_{c+2,1} - s_{c3} + \frac{\alpha(s_{c1} - s_{c2}) + \omega(s_{c+1,1} - s_{c+1,2})}{2},$$

$$v_{c2} = -2B_{c+2,1} + \omega B_{c+3,2} + \alpha B_{c+2,2} + s_{c2} - s_{c1} - \frac{1}{2}(\alpha s_{c3} + \omega s_{c+1,3}),$$

$$B_{c+1,0} = \frac{\alpha B_{c+3,0} - \omega B_{c+2,0}}{\omega^2 + \alpha^2} + \frac{\alpha(s_{c+1,1} + s_{c+1,2}) - \omega(s_{c1} + s_{c2})}{2(\omega^2 + \alpha^2)},$$

$$B_{c+1,1} = \frac{1}{\omega}\left(2B_{c2} + \alpha B_{c1} - B_{c+2,1} - \frac{s_{c1} - s_{c2}}{2}\right),$$

$$B_{c+1,2} = \frac{1}{\omega}\left(-2B_{c1} + \alpha B_{c2} - B_{c+2,2} + \frac{1}{2}s_{c3}\right),$$

$$c = k_3 + 1, k_3 + 3, \cdots, k_4 - 1,$$

$$B_{d0} = \frac{\omega(s_{d+1,1} + s_{d+1,2}) + \alpha(s_{d1} + s_{d2})}{2(\omega^2 + \alpha^2)},$$

$$B_{d1} = \frac{-4\alpha u_{d2} + (\alpha^2 + \omega^2 - 4)u_{d1}}{16\alpha^2 + (\alpha^2 + \omega^2 - 4)^2},$$

$$B_{d2} = \frac{4\alpha u_{d1} + (\alpha^2 + \omega^2 - 4)u_{d2}}{16\alpha^2 + (\alpha^2 + \omega^2 - 4)^2},$$

$$u_{d1} = -s_{d3} + \frac{\alpha(s_{d1} - s_{d2}) + \omega(s_{d+1,1} - s_{d+1,2})}{2},$$

$$u_{d2} = s_{d2} - s_{d1} - \frac{1}{2}(\alpha s_{d3} + \omega s_{d+1,3}),$$

$$B_{d+1,0} = \frac{-\omega(s_{d1} + s_{d2}) + \alpha(s_{d+1,1} + s_{d+1,2})}{2(\omega^2 + \alpha^2)},$$

$$B_{d+1,1} = \frac{1}{\omega}\left(-2B_{d1} + \alpha B_{d2} + \frac{1}{2}s_{d3}\right),$$

$$B_{d+1,2} = \frac{1}{\omega}\left(2B_{d2} + \alpha B_{d1} - \frac{s_{d1} - s_{d2}}{2}\right), d = k_4 + 1,$$

$$k_4 + 3, \cdots, k_5 - 1,$$

$$B_{q0} = \frac{\omega_q(s_{q+1,1} + s_{q+1,2}) + \alpha_q(s_{q1} + s_{q2})}{2(\omega_q^2 + \alpha_q^2)},$$

$$B_{q1} = \frac{-4\alpha_q u_{q2} + (\alpha_q^2 + \omega_q^2 - 4)u_{q1}}{16\alpha_q^2 + (\alpha_q^2 + \omega_q^2 - 4)^2},$$

$$B_{q2} = \frac{4\alpha_q u_{q1} + (\alpha_q^2 + \omega_q^2 - 4)u_{q2}}{16\alpha_q^2 + (\alpha_q^2 + \omega_q^2 - 4)^2},$$

$$u_{q1} = -s_{q3} + \frac{\alpha_q(s_{q1} - s_{q2}) + \omega_q(s_{q+1,1} - s_{q+1,2})}{2},$$

$$u_{q2} = s_{q2} - s_{q1} - \frac{1}{2}(\alpha_q s_{q3} + \omega_q s_{q+1,3}),$$

$$B_{q+1,0} = \frac{-\omega_q(s_{q1} + s_{q2}) + \alpha_q(s_{q+1,1} + s_{q+1,2})}{2(\omega_q^2 + \alpha_q^2)},$$

$$B_{q+1,1} = \frac{1}{\omega_q}\left(-2B_{q1} + \alpha_q B_{q2} + \frac{1}{2}s_{q3}\right),$$

$$B_{q+1,2} = \frac{1}{\omega_q}\left(2B_{q2} + \alpha_q B_{q1} - \frac{s_{q1} - s_{q2}}{2}\right),$$

$$q = k_5 + 1, k_5 + 3, \cdots, n - 1.$$

Proof The normal form of system (5) was given in Refs. [16-17] as follows,

$$\dot{r} = D_2 r + D_4 r + D_6 r + \cdots + D_{2n} r + \cdots$$
$$= a_{13} r^3 + a_{15} r^5 + a_{17} r^7 + \cdots + a_{1(2n+1)} r^{2n+1} + \cdots,$$
$$\dot{\theta} = 1 + D_2 \varphi + D_4 \varphi + D_6 \varphi + \cdots + D_{2n} \varphi + \cdots$$
$$= 1 + a_{23} r^2 + a_{25} r^4 + a_{27} r^6 + \cdots + a_{2(2n+1)} r^{2n} + \cdots.$$
$$(8)$$

Note that the stability of system (5) is the same as system (8).

Hence we need to solve a_{13}, because $a_{13} r^3 = D_2 r$. It is known that the next work is to solve $D_2 r$.

We use perturbation analysis method. We begin by introducing new independent variables according to

$$T_k = \varepsilon^k t, k = 0, 1, 2, \cdots.$$

Then

$$\frac{d}{dt} = \frac{dT_0}{dt}\frac{\partial}{\partial T_0} + \frac{dT_1}{dt}\frac{\partial}{\partial T_1} + \frac{dT_2}{dt}\frac{\partial}{\partial T_2} + \cdots$$
$$= D_0 + \varepsilon D_1 + \varepsilon D_2 + \cdots, \quad (9)$$

where the differential operator $D_k = \partial/\partial T_k$.

Then, suppose that the solution of (5) in the neighborhood of $x = 0$ is represented by an expansion of the form

$$x_i(t;\varepsilon) = \varepsilon x_{i1}(T_0, T_1, \cdots) + \varepsilon^2 x_{i2}(T_0, T_1, \cdots) + \cdots, (i = 1, 2, 3). \quad (10)$$

Note that the perturbation parameter ε used in (10) is the same as that used in the time scales $T_k = \varepsilon^k t (k = 0, 1, 2, \cdots)$. Substituting (9) and (10) into (5) and balancing the like powers of ε results in the ordered perturbation equations. In the ε^1-order perturbation equations, we get

$$D_0^2 x_{11} + x_{11} = 0,$$
$$x_{11} = r(T_1, T_2, \cdots)\cos[T_0 + \varphi(T_1, T_2, \cdots)]$$
$$= r\cos\theta,$$
$$x_{21} = D_0 x_{11} = -r(T_1, T_2, \cdots)\sin[T_0 + \varphi(T_1, T_2, \cdots)] = -r\sin\theta.$$

In the ε^2-order perturbation equation, we find that x_{i2} has the form

$$x_{i2} = B_{i0} + B_{i1}\cos2\theta + B_{i2}\sin2\theta, i = 1, 2, \cdots, n. \quad (11)$$

In the ε^3 -order perturbation equation, by eliminating the possible secular term in x_{13}, we get $D_2 r$. From the first two equations in the ε^3 -order perturbation equations,

$$D_0 x_{13} + D_1 x_{12} + D_2 x_{11} = x_{23} + f_{13},$$
$$D_0 x_{23} + D_1 x_{22} + D_2 x_{21} = - x_{13} + f_{23}, \quad (12)$$

we obtain

$$D_0{}^2 x_{13} + x_{13} = - D_0 D_1 x_{12} - D_0 D_2 x_{11} - D_1 x_{22} - D_2 x_{21} + f_{23} + D_0 f_{13}.$$

Substituting x_{11} and x_{21} into the above equation, we obtain

$$D_0{}^2 x_{13} + x_{13} = 2(D_2 r \cdot \sin\theta + r D_2 \varphi \cdot \cos\theta) + f_{23} + D_0 f_{13}. \quad (13)$$

Note that $f_{i3}(i = 1,2,\cdots,n)$ is in terms of $x_{11}{}^3, x_{21}{}^3$, $x_{11} x_{i2}, x_{21} x_{i2}(i = 1,2,\cdots,n)$. Because $D_2 r$ only appears as the coefficient of $\sin\theta$, we only need to consider the coefficients before $\sin\theta$ in $f_{23} + D_0 f_{13}$. Thus we only consider the terms like $x_{11} x_{i2}, x_{21} x_{i2}$, $x_{11}{}^3(i = 1,2,\cdots,n)$ in f_{13}, and the terms like $x_{11} x_{i2}, x_{21} x_{i2}, x_{21}{}^3(i = 1,2,\cdots,n)$ in f_{23}.

Substituting (11) into (13), we find

$$D_0{}^2 x_{13} + x_{13}$$
$$= 2(D_2 r \cdot \sin\theta + r D_2 \varphi \cdot \cos\theta) +$$
$$\Big[\sum_{i=1}^{n} \Big(\frac{1}{2} \xi_{i2} B_{i2} - \xi_{i1} \Big(B_{i0} + \frac{1}{2} B_{i1} \Big) \Big) +$$
$$\sum_{j=1}^{n} \Big(\frac{1}{2} \tau_{j1} B_{j2} - \tau_{j2} \Big(B_{j0} - \frac{1}{2} B_{j1} \Big) \Big) +$$
$$\frac{1}{2}(B_{22} \xi_{22} + B_{12} \tau_{11}) - \Big(B_{10} + \frac{1}{2} B_{11} \Big) \xi_{11} -$$
$$\Big(B_{20} - \frac{1}{2} B_{21} \Big) \tau_{22} - \frac{3}{4}(s_{16} + s_{26}) \Big] r^3 \sin\theta + g, \quad (14)$$

where g does not contain terms like $\sin\theta$.

To eliminate the possible secular term in x_{13} in the right part of (14), it is required that the coefficients of $\sin\theta$ and $\cos\theta$ equal 0, which in turn yields

$$D_2 r = - \frac{1}{2} r^3 \Big[\sum_{i=1}^{n} \Big(\frac{1}{2} \xi_{i2} B_{i2} - \xi_{i1} \Big(B_{i0} + \frac{1}{2} B_{i1} \Big) \Big) +$$
$$\sum_{j=1}^{n} \Big(\frac{1}{2} \tau_{j1} B_{j2} - \tau_{j2} \Big(B_{j0} - \frac{1}{2} B_{j1} \Big) \Big) +$$
$$\frac{1}{2}(B_{22} \xi_{22} + B_{12} \tau_{11}) - \Big(B_{10} + \frac{1}{2} B_{11} \Big) \xi_{11} -$$
$$\Big(B_{20} - \frac{1}{2} B_{21} \Big) \tau_{22} - \frac{3}{4}(s_{16} + s_{26}) \Big]. \quad (15)$$

Thus the next work is to get B_{i0}, B_{i1}, and B_{i2} and solve x_{i2} in the ε^2 -order perturbation equations. By using the method of harmonic balancing, we get B_{10}, $B_{11}, B_{12}, B_{20}, B_{21}, B_{22}, x_{12}, x_{22}$ and $B_{j0}, B_{j1}, B_{j2}(j = 3, 4,\cdots,n)$ in the ε^2 -order perturbation equations.

Finally, we find that $a_{13} = - \frac{1}{2} \Delta$.

In system (8), if $a_{13} > 0$, $(\Delta < 0)$, then the system is unstable at the origin; if $a_{13} < 0$, $(\Delta > 0)$, then the system is asymptotically stable at the origin. Because systems (8) and (5) have the same stability properties at the origin, thus Theorem 3. 1 is obtained.

Remark Considering the length of this article, a brief proof is given here. More detailed proof and relevant details are given in our other papers and in Refs. [16-17].

Consider a general Hopf bifurcation with forced term expressed in terms of (16),

$$\dot{\boldsymbol{x}} = \boldsymbol{J}(\boldsymbol{x}) + \boldsymbol{F}(\boldsymbol{x}) + \boldsymbol{b}(\boldsymbol{x}) u(\boldsymbol{x}), \quad (16)$$

where $\boldsymbol{J}(\boldsymbol{x})$ and $\boldsymbol{F}(\boldsymbol{x})$ meet the conditions described in (5), $\boldsymbol{b}(\boldsymbol{x}) = (b_1(\boldsymbol{x}), b_2(\boldsymbol{x}), \cdots, b_n(\boldsymbol{x}))^{\mathrm{T}}$, and $u(\boldsymbol{x})$ is a scalar function.

Corollary 3. 1 Suppose that $\boldsymbol{x} = 0$ is an equilibrium point of system (16). If the following conditions hold:

1) Jacobian matrix of system (16) has a pair of purely imaginary eigenvalues $\pm i\omega_c$ at the equilibrium $\boldsymbol{x} = 0$, and the other eigenvalues at the equilibrium $\boldsymbol{x} = 0$ are hyperbolic. In other words, system (16) is the form of Hopf bifurcation,

2) The discriminant Δ of (16) solved based on Theorem 3. 1 is positive. Then $u(\boldsymbol{x})$ can be found for a given $\boldsymbol{b}(\boldsymbol{x})$ to make system (16) asymptotically stable at the origin.

The stability analysis algorithm ensuring the stability of the Hopf bifurcation with forced term is based on Corollary 3. 1 It consists of the steps given as follows.

1) Assume that $u(\boldsymbol{x}) = a_0 + \boldsymbol{a}\boldsymbol{x} + \boldsymbol{x}^{\mathrm{T}} \boldsymbol{B} \boldsymbol{x} + c_1 x_1{}^3 + c_2 x_2{}^3$, where

$\boldsymbol{a} = (a_1, a_2, \cdots, a_n)$, $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)^{\mathrm{T}}$, and $a_0, c_1, c_2, a_i(i = 1,2,\cdots,n)$ are constants, and $\boldsymbol{B}$ is a square matrix of order n.

2) Substitute $u(\boldsymbol{x})$ in (16), and apply the

undetermined coefficient method to determine $u(x)$ by using condition 2) of Corollary 3.1.

3) Check that system (16) with $u(x)$ solved is a Hopf bifurcation, and has an equilibrium $x = 0$. Else go to step 2).

　　The application of this algorithm will be illustrated in section 6.

4　Further explanation for the stability analysis of systems with forced term

　　It is easy to find the form of adding forced item in the actual operation by using our two algorithms. For a general system, we can always find $u(x)$ when $b(x)$ has some certain forms. However, sometimes we can not find $u(x)$ when $b(x)$ is given. This is what we need to explain in this section.

　　Firstly, we show what form $b(x)$ has when $u(x)$ can always be found.

　　Theorem 4.1　　Suppose that $x = 0$ is an equilibrium point of system (1). In system (2), if $b(x)$ is given by $b(x) = Ax$, where A is a positive or negative definite matrix, then we can find $u(x)$ such that system (2) is globally asymptotically stable in the sense of Lyapunov at the origin.

　　Proof　　Based on the algorithm mentioned in section 3, we can prove this theorem easily. We assume that $P = I(I$ is identity matrix) by running step 3) of the algorithm in section 3. Then

$$y = x, \ \dot{V}(x) = 2y^{\mathrm{T}}\dot{y} = 2y^{\mathrm{T}}(f(y) + Ay \cdot v(y)),$$
$$G(y) = 2y^{\mathrm{T}}f(y), C(y) = 2y^{\mathrm{T}}Ay. \text{ If } A \text{ is a positive}$$

definite matrix, then $C(x) > 0$, for $x \neq 0$. Hence we get

$$C^0 = \{y \in Y \mid y = 0\}, C^+ = \{y \in Y \mid y \neq 0\},$$
$$C^- = \phi.$$

　　Then we will find $v(y)$. Because $v(y) \leqslant -\dfrac{G(y)}{C(y)}$ for $y \in C^+$, we determine $v(y)$ by $v(y)$

$$< -\frac{G(y)}{C(y)} = -\frac{y^{\mathrm{T}}f(y)}{y^{\mathrm{T}}Ay} \text{ for } y \in C^+ \text{ so that } v(y)$$

fulfills condition 3) of Theorem 2.1.

　　For example we determine $v(y) = -\dfrac{y^{\mathrm{T}}f(y)}{y^{\mathrm{T}}Ay} - \varepsilon$, where ε is an arbitrary positive number. Then

$$\begin{aligned}
c(y)v(y) &= Ay \cdot v(y) \\
&= A \cdot (y_1 v(y), y_2 v(y), \cdots, y_n v(y))^{\mathrm{T}} \\
&= A\left(y_1\left(-\frac{y^{\mathrm{T}}f(y)}{y^{\mathrm{T}}Ay} - \varepsilon\right),\right. \\
&\quad y_2\left(-\frac{y^{\mathrm{T}}f(y)}{y^{\mathrm{T}}Ay} - \varepsilon\right), \\
&\quad \left.\cdots, y_n\left(-\frac{y^{\mathrm{T}}f(y)}{y^{\mathrm{T}}Ay} - \varepsilon\right)\right).
\end{aligned}$$

We define $A = M^{\mathrm{T}}M, z = My$, where M is an upper triangular matrix in which the diagonal elements are positive. Hence we assume that

$$M^{-1} = \begin{pmatrix} m_{11} & \cdot & \cdot & m_{1n} \\ & \cdot & & \cdot \\ & & \cdot & \cdot \\ & & & m_{nn} \end{pmatrix}.$$

Then, from Cauchy-Buniakowsky-Schwarz inequality it results that:

$$\begin{aligned}
\| (M^{-1}z)^{\mathrm{T}} \| &= \sqrt{(m_{11}z_1 + m_{12}z_2 + \cdots + m_{1n}z_n)^2 + \cdots + (m_{nn}z_n)^2} \\
&\leqslant \sqrt{(m_{11}^2 + \cdots + m_{1n}^2)(z_1^2 + \cdots + z_n^2) + \cdots + m_{nn}^2 z_n^2} \\
&\leqslant \sqrt{n(m_{11}^2 + \cdots + m_{1n}^2)} \sqrt{(z_1^2 + \cdots + z_n^2)}, \\
\Rightarrow & \left| y_i\left(-\frac{y^{\mathrm{T}}f(y)}{y^{\mathrm{T}}Ay} - \varepsilon\right) \right| \\
&= \left| \frac{y_i y^{\mathrm{T}}f(y)}{y^{\mathrm{T}}Ay} + \varepsilon y_i \right| \\
&\leqslant \left| \frac{y_i y^{\mathrm{T}}f(y)}{y^{\mathrm{T}}Ay} \right| + |\varepsilon y_i| \\
&\leqslant \frac{|(m_{ii}z_i + m_{i(i+1)}z_{i+1} + \cdots + m_{in}z_n) \cdot (M^{-1}z)^{\mathrm{T}} \cdot f(y)|}{|z^{\mathrm{T}}z|} + |\varepsilon y_i| \\
&\leqslant \frac{|(m_{ii}z_i + m_{i(i+1)}z_{i+1} + \cdots + m_{in}z_n)| \cdot \|(M^{-1}z)^{\mathrm{T}}\| \cdot \|f(y)\|}{|z^{\mathrm{T}}z|} + |\varepsilon y_i|
\end{aligned}$$

$$\leqslant \frac{\sqrt{m_{ii}^2 + \cdots + m_{in}^2}\ \sqrt{z_i^2 + \cdots + z_n^2}\ \sqrt{n(m_{11}^2 + \cdots + m_{1n}^2)}\ \sqrt{(z_1^2 + \cdots + z_n^2)}\ \sqrt{f_1^2 + \cdots + f_n^2}}{|z^\mathrm{T}z|} + |\varepsilon y_i|$$

$$\leqslant \frac{\sqrt{m_{ii}^2 + \cdots + m_{in}^2}\ \sqrt{n(m_{11}^2 + \cdots + m_{1n}^2)}\ \sqrt{f_1^2 + \cdots + f_n^2}\ |z^\mathrm{T}z|}{|z^\mathrm{T}z|} + |\varepsilon y_i|$$

$$= \sqrt{m_{ii}^2 + \cdots + m_{in}^2}\ \sqrt{n(m_{11}^2 + \cdots + m_{1n}^2)}\ \sqrt{f_1^2 + \cdots + f_n^2} + |\varepsilon y_i|.$$

Hence $c(y)v(y) \to 0$ if $y \to 0$.

In conclusion, $v(y) = -\dfrac{y^\mathrm{T}f(y)}{y^\mathrm{T}Ay} - \varepsilon$ meets all the conditions of Theorem 2.1. Because $y = x$, we find that

$u(x) = -\dfrac{x^\mathrm{T}f(x)}{x^\mathrm{T}Ax} - \varepsilon$, where ε is an arbitrary positive number.

If A is a negative definite matrix, the proof is similar. So we say that we can find $u(x)$ such that system (2) is globally asymptotically stable in the sense of Lyapunov at the origin. The proof is now completed.

Next, we give some counterexamples to show that we can not find $u(x)$ for some $b(x)$.

Counterexample 4. 1 Consider a system in the form

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \tag{17}$$

Suppose that the forced term is given by

$$D(x) = \begin{pmatrix} x_1 \\ -x_2 \end{pmatrix} u(x) = b(x)u(x). \tag{18}$$

Then we add the forced term (18) to system (17), and we get a system with forced term as follows,

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} x_1 \\ -x_2 \end{pmatrix} u(x). \tag{19}$$

We can not find $u(x)$ so that system (19) is globally asymptotically stable in the sense of Lyapunov at the origin. Next we give the proof.

Proof

$$\begin{cases} \dot{x}_1 = x_1 + x_1 u(x) \\ \dot{x}_2 = x_2 - x_2 u(x) \end{cases} \Rightarrow \begin{cases} x_2\dot{x}_1 = x_1x_2 + x_2x_1 u(x) \\ x_1\dot{x}_2 = x_1x_2 - x_1x_2 u(x) \end{cases}$$

$\Rightarrow x_2\dot{x}_1 + x_1\dot{x}_2 = 2x_1x_2 \Rightarrow \dfrac{\dot{x}_1}{x_1} + \dfrac{\dot{x}_2}{x_2} = 2 \Rightarrow \ln x_1 + \ln x_2 = 2t + c \Rightarrow \ln x_1 x_2 = 2t + c \Rightarrow x_1 x_2 = $

$$\mathrm{e}^{2t+c} = C\mathrm{e}^{2t}, \tag{20}$$

where c and C are constants. Hence we get that $x_1 x_2$ $= C\mathrm{e}^{2t}$. If system (19) is globally asymptotically stable in the sense of Lyapunov at the origin, then $x_1(t) \to 0$, $x_2(t) \to 0$, as $t \to \infty$. However, $x_1x_2 = C\mathrm{e}^{2t} \to \infty$, as $t \to \infty$.

So a contradiction appears. The proof is now complete.

Counterexample 4. 2 Consider a system in the form (17), suppose that the forced term is given by

$$D(x) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} u(x) = b(x)u(x). \tag{21}$$

Then we add the forced term (21) to system (17), and we get a system with forced term as follows,

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} u(x). \tag{22}$$

We can not find $u(x)$ so that system (22) is globally asymptotically stable in the sense of Lyapunov at the origin. Next we give the proof.

Proof

$$\begin{cases} \dot{x}_1 = x_1 + u(x) \\ \dot{x}_2 = x_2 - u(x) \end{cases} \Rightarrow \dot{x}_1 + \dot{x}_2 = x_1 + x_2 \Rightarrow$$

$$\frac{\dot{x}_1 + \dot{x}_2}{x_1 + x_2} = 1 \Rightarrow \ln(x_1 + x_2) = t + c \Rightarrow$$

$$x_1 + x_2 = \mathrm{e}^{t+c} = C\mathrm{e}^t, \tag{23}$$

where c and C are constants. Hence we get that $x_1 + x_2 = C\mathrm{e}^t$. If system (19) is globally asymptotically stable in the sense of Lyapunov at the origin, then $x_1(t) \to 0$, $x_2(t) \to 0$, as $t \to \infty$. However, $x_1 + x_2 = C\mathrm{e}^t \to \infty$, as $t \to \infty$.

So a contradiction appears. The proof is now complete.

5 Some examples

This section is dedicated to the validation of the theoretical results derived in sections 3, 4, and 5.

Firstly, we give a nonlinear system with forced term, the inverted pendulum on a cart system. This

simple mechanical system is representative to model a class of attitude control problems, and the goal is to maintain permanently the desired vertically oriented position. Since the inverted pendulum is a nonlinear system, the basic balance equations for the system are derived firstly and put into the standard state-space form. Given an inverted pendulum mounted on a cart as shown in Fig. 1, the first principle nonlinear equations are applied in the sequel. Assuming that the rod is massless and that the cart mass and the point mass at the upper end of the inverted pendulum are denoted as M and m, respectively, the gravity force acts on the point mass at all times. The coordinate system is defined in Fig. 1, where $x(t)$ represents the cart position and $\theta(t)$ is the tilt angle with respect to the vertical upward direction.
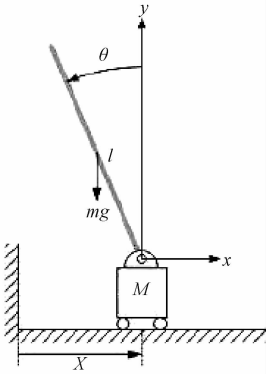

Fig. 1   Variables related to the inverted pendulum on a cart system

The differential equation that describes the behavior of the simple system is usually written as

$$(m + M) \cdot l^2 \cdot \ddot{\theta} - (m + M) \cdot l \cdot g \cdot \sin(\theta) = 0, \tag{24}$$

where M is the mass of the cart, m is the mass of the pendulum, l is the length of pendulum (distance to the center of mass), x is the cart position coordinate, and θ is the pendulum angle with respect to the vertical position.

The state vector consists of the angle θ and the angular velocity of the pendulum $\dot{\theta}$. Therefore, the two state variables are defined as z_1 and z_2, where $z_1 \in [-80, 80], z_2 \in [-30, 30], z_1(t) = \theta(t)$, $z_2(t) = \dot{\theta}(t)$. In order to write equation (24) in terms of state variables, they are substituted

resulting in

$$\dot{z} = f(z), \tag{25}$$

where $z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ – state vector, $f(z) = \begin{pmatrix} z_2 \\ \dfrac{g}{l}\sin z_1 \end{pmatrix}$.

Obviously, system (25) is unstable in the sense of Lyapunov at the origin $z = 0$. In other words, we can not ensure the upright stabilization of the pendulum aiming at the setpoint value of z, $z = 0$. In order to make the inverted pendulum on the cart system asymptotically stable, we add a forced term to system (25), and the system with forced term is expressed in the form

$$\dot{z} = f(z) + b(z)u(z), \tag{26}$$

where $z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ – state vector,

$$f(z) = \begin{pmatrix} z_2 \\ \dfrac{g}{l}\sin z_1 \end{pmatrix}, \quad b(z) = \begin{pmatrix} 0 \\ -\dfrac{1}{(m + M)l^2} \end{pmatrix}.$$

Note that $b(z)$ is given according to the physical meaning.

The algorithm presented in section 3 will be applied as follows, in order to find the value of $u(x)$ at which system (26) is stabilized.

Step 1: Give a general upper triangular matrix in which the diagonal elements are positive,

$$Q = \begin{pmatrix} a_{11} & a_{12} \\ & a_{22} \end{pmatrix}; \quad a_{11}, a_{22} > 0.$$

Step 2: Transform system (26) to system (27) by the linear transformation $y = Qx$, and get the expressions of $G(y), C(y)$, and C^0,

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} \dfrac{a_{11}}{a_{22}}y_2 + \dfrac{ga_{12}}{l}\sin\left(\dfrac{1}{a_{11}}y_1 - \dfrac{a_{12}}{a_{11}a_{22}}y_2\right) \\ \dfrac{ga_{22}}{l}\sin\left(\dfrac{1}{a_{11}}y_1 - \dfrac{a_{12}}{a_{11}a_{22}}y_2\right) \end{pmatrix} +$$

$$\begin{pmatrix} -\dfrac{a_{12}}{(m + M)l^2} \\ -\dfrac{a_{22}}{(m + M)l^2} \end{pmatrix} v(y), \tag{27}$$

$$\dot{V}(z) = 2(y_1\dot{y}_1 + y_2\dot{y}_2)$$

$$= 2\left[\dfrac{a_{11}}{a_{22}}y_1y_2 + \left(\dfrac{g}{l}a_{12}y_1 + \dfrac{g}{l}a_{22}y_2\right)\right.$$

$$\sin\left(\frac{1}{a_{11}}y_1 - \frac{a_{12}}{a_{11}a_{22}}y_2\right)\Big] -$$

$$\frac{2}{(m + M)l^2}(y_1 a_{12} + y_2 a_{22})v(\boldsymbol{y}), \qquad (28)$$

$$G(\boldsymbol{y}) = 2\Big[\frac{a_{11}}{a_{22}}y_1 y_2 + \Big(\frac{g}{l}a_{12}y_1 + \frac{g}{l}a_{22}y_2\Big) \cdot$$
$$\sin\left(\frac{1}{a_{11}}y_1 - \frac{a_{12}}{a_{11}a_{22}}y_2\right)\Big], \qquad (29)$$

$$C(\boldsymbol{y}) = -\frac{2}{(m + M)l^2}(y_1 a_{12} + y_2 a_{22}),$$

$$C^0 = \left\{\boldsymbol{y} \in Y \mid y_2 = -\frac{a_{12}}{a_{22}}y_1\right\}, \qquad (30)$$

where $Y = [-80a_{11} - 30a_{12}, 80a_{11} + 30a_{12}] \times [-30a_{22}, 30a_{22}]$.

Step 3: Apply the undetermined coefficient method to determine the elements of $\boldsymbol{Q}$ by using condition 1) of Theorem 2.1. The condition is that $G(\boldsymbol{y}) \leqslant 0$ for $\forall \boldsymbol{y} \in C^0$. Next, we substitute $y_2 = -\frac{a_{12}}{a_{22}}y_1$ into (29) and get

$$G(\boldsymbol{y}) = 2\Big[\frac{a_{11}}{a_{22}}y_1 y_2 + \Big(\frac{g}{l}a_{12}y_1 + \frac{g}{l}a_{22}y_2\Big) \cdot$$
$$\sin\left(\frac{1}{a_{11}}y_1 - \frac{a_{12}}{a_{11}a_{22}}y_2\right)\Big]$$
$$= 2\Big[-\frac{a_{11}a_{12}}{a_{22}{}^2}y_1{}^2 + \frac{g}{l}(a_{12}y_1 - a_{12}y_1) \cdot$$
$$\sin\left(\frac{1}{a_{11}}y_1 + \frac{a_{12}{}^2}{a_{11}a_{22}{}^2}y_1\right)\Big]$$
$$= -\frac{2a_{11}a_{12}}{a_{22}{}^2}y_1{}^2. \qquad (31)$$

In order to decrease the complexity, we assume $a_{11} = a_{22} = 1, a_{12} = 0$ so that $G(\boldsymbol{y}) \leqslant 0$ for $\forall \boldsymbol{y} \in C^0$. Next, we substitute $a_{11} = a_{22} = 1, a_{12} = 0$ into (28), (29), and (30),

$$\dot{V}(\boldsymbol{z}) = 2\big(y_1\dot{y}_1 + y_2\dot{y}_2\big) = 2\big(y_1 y_2 + \frac{g}{l}y_2\sin y_1\big) -$$
$$\frac{2}{(m + M)l^2}y_2 v(\boldsymbol{y}), \qquad (32)$$

$$C(\boldsymbol{y}) = -\frac{2}{(m + M)l^2}y_2, G(\boldsymbol{y}) = 2\big(y_1 y_2 +$$
$$\frac{g}{l}y_2\sin y_1\big), \qquad (33)$$

$$C^0 = \{\boldsymbol{y} \in Y \mid y_2 = 0\},$$

and we also get

$$C^+ = \{\boldsymbol{y} \in Y \mid y_2 < 0\}, C^- = \{\boldsymbol{y} \in Y \mid y_2 > 0\},$$

where $Y = [-80, 80] \times [-30, 30]$.

Step 4: Determine $v(\boldsymbol{y})$ so that

$$v(\boldsymbol{y}) \leqslant -\frac{G(\boldsymbol{y})}{C(\boldsymbol{y})} = l(m + M)(y_1 l + g\sin y_1) \text{ for } \boldsymbol{y} \in C^+,$$

$$v(\boldsymbol{y}) \geqslant -\frac{G(\boldsymbol{y})}{C(\boldsymbol{y})} = l(m + M)(y_1 l + g\sin y_1) \text{ for } \boldsymbol{y} \in C^-,$$

$c(\boldsymbol{y})v(\boldsymbol{y}) \to 0$ for $\boldsymbol{y} \to 0$. Thus, it can be taken

$$v(\boldsymbol{y}) = l(m + M)(y_1 l + g\sin y_1) + y_2. \quad (34)$$

Step 5: Check that $v(\boldsymbol{y})$ fulfills condition 3) of Theorem 2.1. By substituting (34) into (32), the derivative is

$$\dot{V}(\boldsymbol{z}) = 2y_2\Big(y_1 + \frac{g}{l}\sin y_1 - \frac{1}{(m + M)l^2}v(\boldsymbol{y})\Big)$$
$$= -\frac{2}{(m + M)l^2}y_2{}^2,$$

with $\dot{V}(0) = 0$. Assume that there is a trajectory with $y_2(t) = 0$ and $y_1(t) \neq 0$. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}y_2(t) = \frac{g}{l}\sin(y_1(t)) - \frac{1}{(m + M)l^2}v(\boldsymbol{y})$$
$$= \frac{g}{l}\sin(y_1(t)) - \frac{1}{(m + M)l^2} \cdot$$
$$[l(m + M)(y_1 l + g\sin y_1(t)) + y_2]$$
$$= -y_1 - \frac{1}{(m + M)l^2}y_2 \neq 0, \qquad (35)$$

which means that $y_2(t)$ can not stay constant. Hence, $\boldsymbol{y}(t) = 0$ is the only possible state trajectory for which $\dot{V}(\boldsymbol{z}) = 0$. So $v(\boldsymbol{y})$ fulfills condition 3) of Theorem 2.1. The set $\{\boldsymbol{y} \in Y \mid \Omega = 0\}$ contains no state trajectories except the trivial one, $\boldsymbol{y}(t) = 0$, $t \geqslant 0$.

Step 6: $v(\boldsymbol{y})$ has been obtained. Find $u(\boldsymbol{z})$ by using the linear transformation $\boldsymbol{y} = \boldsymbol{Q}\boldsymbol{z}$, $(\boldsymbol{y} = \boldsymbol{z})$. Thus, $u(\boldsymbol{z}) = l(m + M)(z_1 l + g\sin z_1) + z_2$.

To cnclude, the system with forced term (26) designed here is globally asymptotically stable in the sense of Lyapunov at the origin. Note that $u(\boldsymbol{z})$ is equal to the externally x-directed force, $u = F$, which is presented in Fig. 2.

Next, we consider a system with Hopf bifurcation in the form

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} x_2 \\ -x_1 \\ -x_3 \end{pmatrix} + \begin{pmatrix} x_1{}^2 + x_1 x_2 \\ x_1{}^2 \\ 0 \end{pmatrix}. \qquad (36)$$

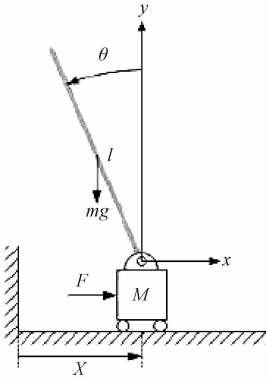For system (36), we find that $\Delta = -1 < 0$. Thus

Fig. 2　Variables related to the inverted pendulum on a cart system with forced term

we can know that system (36) is unstable according to Theorem 3.1. Then we will add a forced term to system (36) so that the new system with the forced term is asymptotically stable at the origin. We assume that the new system with forced term is expressed in the form

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} x_2 \\ -x_1 \\ -x_3 \end{pmatrix} + \begin{pmatrix} x_1^2 + x_1 x_2 \\ x_1^2 \\ 0 \end{pmatrix} + \begin{pmatrix} b_1(\boldsymbol{x}) \\ b_2(\boldsymbol{x}) \\ b_3(\boldsymbol{x}) \end{pmatrix} u(\boldsymbol{x}).$$

(37)

Without loss of generality, for simplifying the calculation, we assume that $b_i(\boldsymbol{x})$, $(i = 1,2,3)$ here is given as constant. Thus system (37) is rewritten as follows,

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} x_2 \\ -x_1 \\ -x_3 \end{pmatrix} + \begin{pmatrix} x_1^2 + x_1 x_2 \\ x_1^2 \\ 0 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} u(\boldsymbol{x}). \quad (38)$$

We next find $u(\boldsymbol{x})$ so that system (38) is asymptotically stable at the origin by our algorithm in section 4. Because the system is simple, we can assume that $u(\boldsymbol{x}) = a x_1^2$ (a is constant) and substitute $u(\boldsymbol{x})$ in (38). We apply the undetermined coefficient method to determine $u(\boldsymbol{x})$ using condition 2) of Corollary 3.1. Based on the calculation, we know that system (38) will be asymptotically stable at the origin, if the following condition holds：

$$(2b_2 a + 1)(b_1 a + 1) < 0. \quad (39)$$

Thus, if $b_i = 1$, $(i = 1,2,3)$, we can find $u(\boldsymbol{x})$ $= -\dfrac{2}{3} x_1^2$ so that system (38) is asymptotically

stable at the origin.

6　Conclusions

We first introduce the system with forced term. Next, an approach to the global asymptotically stability analysis of general system with forced term is proposed. The discrimination method for an n-dimensional system stability is given, and an algorithm to the asymptotically stability analysis of Hopf bifurcation with forced term is developed beased on this discrimination method. Next, a theorem which shows what form of forced term can certainly be found. Some counterexamples, which show that some forms of forced term can not be found for a general system, are given. Some examples show how the stability analysis algorithm suggested here can be applied to systems with forced term. In this study, we give two methods for adding the forced term in theory. Scholars with a wealth of physical knowledge background can further study some specific forms of forced term, which are easy to add for the actual physical problems. Further we will prove that the stability approach in this work can be applied also in situation where the system has an equilibrium point different from the origin.

References

[1] Leine R I. The historical development of classical stability concepts：Lagrange, Poisson and Lyapunov stability [J]. Nonlinear Dynamics, 2010, 59(1/2)：173-182.

[2] Bălaş M M, Bălaş V E. World Knowledge for control applications by fuzzy-interpolative systems[J]. International Journal of Computers Communications & Control, 2008, 3 (1)：28-32.

[3] Bazoula A, Djouadi M S, Maaref H. Formation control of multi-robots via fuzzy logic technique [J]. International Journal of Computers Communications & Control, 2008, 3 (2)：179-184.

[4] Bellomo D, Naso D, Babuška R. Adaptive fuzzy control of a non-linear servo-drive：theory and experimental results[J]. Engineering Applications of Artificial Intelligence, 2008, 21 (6)：846-857.

[5] Mohan B M, Sinha A. Analytical structure and stability analysis of a fuzzy PID controller [J]. Applied Soft Computing, 2008, 8(1)：749-758.

[6] Perry A G, Feng G, Liu Y F, et al. A design method for PI-like fuzzy logic controllers for DC-DC converter[J]. IEEE

Transactions on Industrial Electronics, 2007, 54(5): 2 688-2 696.

[7] Vesselenyi T, Dziţac S, Dziţac I, et al. Fuzzy and neural controllers for a pneumatic actuator[J]. International Journal of Computers Communications & Control, 2007, 2(4): 375-387.

[8] Feng G. A survey on analysis and design of model-based fuzzy control systems[J]. IEEE Transactions on Fuzzy systems, 2006, 14(5): 676-697.

[9] Michels K, Klawonn F, Kruse R, et al. Fuzzy control: fundamentals, stability and design of fuzzy controllers[M]. New York: Springer, 2007.

[10] Precup R E, Preitl S, Rudas I J, et al. Design and experiments for a class of fuzzy controlled servo systems[J]. IEEE/ASME Transactions on Mechatronics, 2008, 13(1): 22-35.

[11] Tian E, Peng C. Delay-dependent stability analysis and synthesis of uncertain T-S fuzzy systems with time-varying delay[J]. Fuzzy Sets and Systems, 2006, 157(4): 544-559.

[12] Khalil H K, Grizzle J W. Nonlinear systems[M]. New Jersey: Prentice Hall, 1996.

[13] Haddad W M, Chellaboina V S. Nonlinear dynamical systems and control: a Lyapunov-based approach[M]. Princeton: Princeton University Press, 2008.

[14] Vidyasagar M. Nonlinear systems analysis[M]. Philadelphia: Society for Industrial & Applied Mathematics (SIAM), 2002.

[15] LaSalle J P. Some extensions of Liapunov's second method [J]. Circuit Theory, IRE Transactions on Circuit Theory, 1960, 7(4): 520-527.

[16] Yu P. Simplest Normal Forms of Hopf and Generalized Hopf Bifurcations [J]. International Journal of Bifurcation & Chaos, 2011, 9(10):1 917-1 939.

[17] Yu P. Computation of normal forms via a perturbation technique[J]. Journal of Sound and Vibration, 1998, 211 (1): 19-38.