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Herz space with variable exponent on spaces of homogeneous type*

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Abstract In this work, a certain Herz space with variable exponent on spaces of homogeneous type is defined, and the block decomposition for this space is established. Using this decomposition, some boundedness for a class of sublinear operators on Herz space with variable exponent on spaces of homogeneous type is obtained.

Keywords Herz space; variable exponent; spaces of homogeneous type; block decomposition; boundedness

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齐型空间上的变指标 Herz 空间

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摘　要　引入一类齐型空间上的变指标 Herz 空间,并建立该空间的块分解. 利用此分解得到一类次线性算子在上述变指标 Herz 空间中的一些有界性.

关键词　Herz 空间; 变指标; 齐型空间; 块分解; 有界性

The theory of function spaces with variable exponent has been extensively studied by researchers since the work of Kováčik and Rákosník[1] appeared in 1991 (See Refs. [2-4] and references therein). Inspired by Refs. [5-7], we introduce the Herz space with variable exponent on spaces of homogeneous type and obtain the block decomposition for them. Meanwhile, we obtain some boundedness for a class of sublinear operators on the Herz space with variable exponent on spaces of homogeneous type, using this decomposition.

Firstly we give some notations and basic definitions on variable Lebesgue spaces on spaces of homogeneous type. Let $X=(X,d,\mu)$ be a space of homogeneous type in the sense of Coifman and Weiss[8]. This is a topological space X endowed with

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a Borel measure μ and a quasi-metric (or quasi-distance) d. The latter is a mapping $d: X \times X \to \mathbb{R}^+ = \{t \in \mathbb{R}: t \geqslant 0\}$ satisfying

(i) $d(x,y) = d(y,x)$,

(ii) $d(x,y) > 0$ if and only if $x \neq y$,

(iii) there exists a constant K such that $d(x,y) \leqslant K[d(x,z) + d(z,y)]$ for all x, y, z in X.

We postulate that $\mu(B(x,r)) > 0$ whenever $r > 0$, where $B(x,r) = \{y \in X: d(x,y) < r\}$ denotes the open ball centered at x with a radius r. Our basic assumption relating the measure and the quasi-distance is the existence of a constant A such that

$$\mu(B(x,2r)) \leqslant A\mu(B(x,r)). \tag{1}$$

As known (See Lemma 14.6 in Ref. [9]), from (1) there follows the property

$$\frac{\mu(B(x,R))}{\mu(B(y,r))} \leqslant A\left(\frac{R}{r}\right)^N,\ N = \log_2 A, \tag{2}$$

for all the balls $B(x,R)$ and $B(y,r)$ with $0 < r \leqslant R$ and $y \in B(x,r)$. From (2) we have

$$\mu(B(x,r)) \geqslant Cr^N,\ x \in \Omega,\ 0 < r \leqslant l, \tag{3}$$

for any $l < +\infty$ and any open set $\Omega \subset X$ on which $\inf_{x\in\Omega}\mu(B(x,l)) > 0$. Condition (3) is also known as the lower Ahlfors regularity condition.

In addition, the space of homogeneous type (X,d,μ) is assumed in Ref. [5] to satisfy the conditions

(i) $\mu(\{x\}) = 0$, $\mu(X) = \infty$,

(ii) there exist constants $a \geqslant 2$ and $A_0 > 1$ such that

$$\mu(B(x,ar)) \geqslant A_0\mu(B(x,r)) \tag{4}$$

holds for all $x \in X$ and $0 < r < \infty$.

Given a μ-measurable function $p: X \to [1,\infty)$, $L^{p(\cdot)}(X)$ denotes the set of μ-measurable functions f on X such that for some $\lambda > 0$,

$$\int_X \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} d\mu(x) < \infty.$$

This set becomes a Banach function space when equipped with the Luxemburg-Nakano norm

$$\|f\|_{L^{p(\cdot)}(X)} = \inf\left\{\lambda > 0: \int_X \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} d\mu(x) \leqslant 1\right\}.$$

These spaces are referred to as variable Lebesgue spaces on spaces of homogeneous type.

The space $L^{p(\cdot)}_{\mathrm{loc}}(X)$ is defined by $L^{p(\cdot)}_{\mathrm{loc}}(X) := \{f: f \in L^{p(\cdot)}(E)$ for all compact subsets $E \subset X\}$. Define $P(X)$ to be set of $p(\cdot): X \to [1,\infty)$ such that

$$p^- = \operatorname{ess\,inf}\{p(x): x \in X\} > 1,$$
$$p^+ = \operatorname{ess\,sup}\{p(x): x \in X\} < \infty.$$

Denote $p'(x) = p(x)/(p(x)-1)$.

Next we recall some basic properties of the spaces $L^{p(\cdot)}(X)$. The Hölder inequality is valid in the form

$$\int_X |f(x)g(x)|\, d\mu(x) \leqslant r_p \|f\|_{L^{p(\cdot)}} \|g\|_{L^{p'(\cdot)}}, \tag{5}$$

where

$$r_p = 1 + 1/p^- - 1/p^+.$$

The standard local logarithmic condition on X is usually introduced in the form

$$|p(x) - p(y)| \leqslant \frac{A_p}{-\ln d(x,y)},$$
$$d(x,y) \leqslant 1/2,\ x, y \in X, \tag{6}$$

where $A_p > 0$ is independent of x and y. Condition (6) is known as Dini-Lipschitz condition or log-Hölder continuity condition.

1 Main results and their proofs

In this section, firstly we give the definition of the Herz space with variable exponent on spaces of homogeneous type (X,d,μ). Let $x_0 \in X$, $B_k = \{x \in X: d(x_0,x) < a^k\}$, and $R_k = B_k \backslash B_{k-1}$ for $k \in \mathbb{Z}$. Denote $\mathbb{Z}_+$ and $\mathbb{N}$ as the sets of all positive and non-negative integers, $\chi_k = \chi_{R_k}$ for $k \in \mathbb{Z}$, $\tilde{\chi}_k = \chi_k$ if $k \in \mathbb{Z}_+$, and $\tilde{\chi}_0 = \chi_{B_0}$, where χ_{R_k} is the characteristic function of R_k.

Definition 1.1 Let $0 < \alpha < \infty$, $0 < p < \infty$, and $q(\cdot) \in P(X)$. The homogeneous Herz space with variable exponent $\dot{K}^{\alpha,p}_{q(\cdot)}(X)$ on spaces of homogeneous type (X,d,μ) is defined by

$$\dot{K}^{\alpha,p}_{q(\cdot)}(X) = \{f \in L^{q(\cdot)}_{\mathrm{loc}}(X\backslash\{x_0\}): \|f\|_{\dot{K}^{\alpha,p}_{q(\cdot)}(X)} < \infty\},$$

where

$$\|f\|_{\dot{K}^{\alpha,p}_{q(\cdot)}(X)} = \left\{\sum_{k\in\mathbb{Z}} \mu(B_k)^{\alpha p} \|f\chi_k\|^p_{L^{q(\cdot)}(X)}\right\}^{1/p}.$$

The non-homogeneous Herz space with variable exponent $K^{\alpha,p}_{q(\cdot)}(X)$ on spaces of homogeneous type (X,d,μ) is defined by

$$K_{q(\cdot)}^{\alpha,p}(X)=\{f\in L_{\mathrm{loc}}^{q(\cdot)}(X):\|f\|_{K_{q(\cdot)}^{\alpha,p}(X)}<\infty\},$$
where
$$\|f\|_{K_{q(\cdot)}^{\alpha,p}(X)}=\Big\{\sum_{k=0}^{\infty}\mu(B_k)^{\alpha p}\|f\tilde{\chi}_k\|_{L^{q(\cdot)}(X)}^p\Big\}^{1/p}.$$

Remark 1.1 When $X=\mathbb{R}^n$, $d(x,y)=|x-y|=(\sum_{j=1}^{n}(x_j-y_j)^2)^{1/2}$, $x_0=0$, and μ equals Lebesgue measure, we have
$$\dot{K}_{q(\cdot)}^{\alpha,p}(X)=\dot{K}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)$$
and
$$K_{q(\cdot)}^{\alpha,p}(X)=K_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n),$$
where $\dot{K}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)$ and $K_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)$ were introduced by Izuki in Ref. [10].

Next, we consider the decomposition of $\dot{K}_{q(\cdot)}^{\alpha,p}(X)$. We begin with the notation of central block.

Definition 1.2 Let $0<\alpha<\infty$ and $q(\cdot)\in P(X)$.

(i) A function $a(x)$ on X is said to be a central $(\alpha,q(\cdot))$-block if

(a) supp $a\subset B_k$.

(b) $\|a\|_{L^{q(\cdot)}(X)}\leqslant\mu(B_k)^{-\alpha}$

(ii) A fuction $a(x)$ on X is said to be a central $\alpha,q(\cdot)$-block of restricted type if

(a) supp $a\subset B_k$ for some $1\leqslant d(x_0,x)<a^k$.

(b) $\|a\|_{L^{q(\cdot)}(X)}\leqslant\mu(B_k)^{-\alpha}$.

The decomposition theorem (See below) shows that the central blocks are the "building block" of the Herz space with variable exponent on spaces of homogeneous type.

Theorem 1.1 Let $0<\alpha<\infty$, $0<p<\infty$, and $q(\cdot)\in P(X)$. The following two statements are equivalent:

(i) $f\in\dot{K}_{q(\cdot)}^{\alpha,p}(X)$;

(ii) f can be represented by
$$f(x)=\sum_{k\in\mathbb{Z}}\lambda_k b_k(x),\tag{7}$$
where each b_k is a central $(\alpha,q(\cdot))$-block with support contained in B_k and $\sum_k|\lambda_k|^p<\infty$.

Proof We first prove that (i) implies (ii). For every $f\in\dot{K}_{q(\cdot)}^{\alpha,p}(X)$, write
$$f(x)=\sum_{k\in\mathbb{Z}}f(x)\chi_k(x)$$
$$=\sum_{k\in\mathbb{Z}}\mu(B_k)^{\alpha}\|f\chi_k\|_{L^{q(\cdot)}(X)}\times\frac{f(x)\chi_k(x)}{\mu(B_k)^{\alpha}\|f\chi_k\|_{L^{q(\cdot)}(X)}}$$
$$=\sum_{k\in\mathbb{Z}}\lambda_k b_k(x),$$
where $\lambda_k=\mu(B_k)^{\alpha}\|f\chi_k\|_{L^{q(\cdot)}(X)}$ and $b_k(x)=\dfrac{f(x)\chi_k(x)}{\mu(B_k)^{\alpha}\|f\chi_k\|_{L^{q(\cdot)}(X)}}$.

It is obvious that supp $b_k\subset B_k$ and $\|b_k\|_{L^{q(\cdot)}(X)}=\mu(B_k)^{-\alpha}$. Thus, each b_k is a central $(\alpha,q(\cdot))$-block with the support B_k and
$$\sum_{k\in\mathbb{Z}}|\lambda_k|^p=\sum_{k\in\mathbb{Z}}\mu(B_k)^{\alpha p}\|f\chi_k\|_{L^{q(\cdot)}(X)}^p$$
$$=\|f\|_{\dot{K}_{q(\cdot)}^{\alpha,p}(X)}^p<\infty.$$

Now we prove that (ii) implies (i). Let $f(x)=\sum_{k\in\mathbb{Z}}\lambda_k b_k(x)$ be a decomposition of f which satisfies hypothesis (ii) of Theorem 1.1. For each $j\in\mathbb{Z}$, by Minkowski inequality, we have
$$\|f\chi_j\|_{L^{q(\cdot)}(X)}\leqslant\sum_{k\geqslant j}|\lambda_k|\ \|b_k\|_{L^{q(\cdot)}(X)}.\tag{8}$$
Now we consider two cases for the index p. If $0<p\leqslant1$, from (8) it follows that
$$\begin{aligned}&\|f\|_{\dot{K}_{q(\cdot)}^{\alpha,p}(X)}^p\\&=\sum_{k\in\mathbb{Z}}\mu(B_k)^{\alpha p}\|f\chi_k\|_{L^{q(\cdot)}(X)}^p\\&\leqslant\sum_{k\in\mathbb{Z}}\mu(B_k)^{\alpha p}\Big(\sum_{j\geqslant k}|\lambda_k|^p\|b_j\|_{L^{q(\cdot)}(X)}^p\Big)\\&\leqslant\sum_{k\in\mathbb{Z}}\mu(B_k)^{\alpha p}\Big(\sum_{j\geqslant k}|\lambda_k|^p\mu(B_j)^{-\alpha p}\Big)\\&=\sum_{k\in\mathbb{Z}}|\lambda_k|^p\sum_{j\geqslant k}\left(\frac{\mu(B_k)}{\mu(B_j)}\right)^{\alpha p}\\&\leqslant\sum_{k\in\mathbb{Z}}|\lambda_k|^p\sum_{j\geqslant k}A_0^{(k-j)\alpha p}\\&\leqslant C\sum_{k\in\mathbb{Z}}|\lambda_k|^p.\end{aligned}$$

If $1<p<\infty$, by (8) and the Hölder inequality we have
$$\begin{aligned}&\|f\chi_j\|_{L^{q(\cdot)}(X)}\\&\leqslant\sum_{k\geqslant j}|\lambda_k|\ \|b_k\|_{L^{q(\cdot)}(X)}^{1/2}\|b_k\|_{L^{q(\cdot)}(X)}^{1/2}\\&\leqslant\Big(\sum_{k\geqslant j}|\lambda_k|^p\|b_k\|_{L^{q(\cdot)}(X)}^{p/2}\Big)^{1/p}\times\\&\quad\Big(\sum_{k\geqslant j}\|b_k\|_{L^{q(\cdot)}(X)}^{p'/2}\Big)^{1/p'}\\&\leqslant\Big(\sum_{k\geqslant j}|\lambda_k|^p\mu(B_k)^{-\alpha p/2}\Big)^{1/p}\times\Big(\sum_{k\geqslant j}\mu(B_k)^{-\alpha p'/2}\Big)^{1/p'}.\end{aligned}$$
Therefore,
$$\begin{aligned}&\|f\|_{\dot{K}_{q(\cdot)}^{\alpha,p}(X)}^p\\&\leqslant C\sum_{j\in\mathbb{Z}}\mu(B_j)^{\alpha p}\Big(\sum_{k\geqslant j}|\lambda_k|^p\mu(B_k)^{-\alpha p/2}\Big)\times\end{aligned}$$

$$\Big(\sum_{k\geqslant j}\mu(B_k)^{-\alpha p'/2}\Big)^{p/p'}$$
$$\leqslant C\sum_{k\in\mathbb{Z}}|\lambda_k|^p\sum_{j\leqslant k}A_0^{\alpha(j-k)p/2}$$
$$\leqslant C\sum_{k\in\mathbb{Z}}|\lambda_k|^p.$$

This leads to $f\in\dot K_{q(\cdot)}^{\alpha,p}(X)$ and then completes the proof of Theorem 1.1.

Remark 1.2 From the proof of Theorem 1.1, it is easy to see that if $f\in\dot K_{q(\cdot)}^{\alpha,p}(X)$ and $f(x)=\sum_{k\in\mathbb{Z}}\lambda_k b_k(x)$ is a central $(\alpha,q(\cdot))$-block decomposition, then

$$\|f\|_{\dot K_{q(\cdot)}^{\alpha,p}(X)}\approx\Big(\sum_{k\in\mathbb{Z}}|\lambda_k|^p\Big)^{1/p}.$$

By an argument similar to the proof of Theorem 1.1, we can obtain the decomposition characterizations of the non-homogeneous Herz space with variable exponent of X as follows.

Theorem 1.2 Let $0<\alpha<\infty$, $0<p<\infty$, and $q(\cdot)\in P(X)$. The following two statements are equivalent:

(i) $f\in K_{q(\cdot)}^{\alpha,p}(X)$;

(ii) f can be represented by

$$f(x)=\sum_{k=0}^{\infty}\lambda_k b_k(x),\tag{9}$$

where each b_k is a central $(\alpha,q(\cdot))$-block of restricted type with support contained in B_k and $\sum_{k\geqslant 0}|\lambda_k|^p<\infty$.

Moreover, the norms $\|f\|_{K_{q(\cdot)}^{\alpha,p}}(X)$ and $\inf\Big(\sum_{k\geqslant 0}|\lambda_k|^p\Big)^{1/p}$ are equivalent, where the infimum is taken over all decompositions of f as in (9).

As applications of the decomposition theorems, let us come to investigate the boundedness for some sublinear operators on the Herz space with variable exponent on X.

Theorem 1.3 Let X be bounded, $q(\cdot)\in P(X)$ satisties condition (6), $0<p<\infty$, and $0<\alpha<1-\frac{1}{q^-}$. If a sublinear operator T satisfies

$$|Tf(x)|\leqslant C\|f\|_{L^1(X)}/\mu(B(x_0,\mathrm{d}(x_0,x))),$$
$$\text{if dist}\,(x,\operatorname{supp}f)>\frac{\mathrm{d}(x_0,x)}{2K},\tag{10}$$

for any integrable function f with a compact support and T is bounded on $L^{q(\cdot)}(X)$, then T is bounded on $\dot K_{q(\cdot)}^{\alpha,p}(X)$ and $K_{q(\cdot)}^{\alpha,p}(X)$, respectively.

To prove Theorem 1.3, we need the follwing auxiliary result.

Lemma 1.1[11] Let X be bounded, the measure μ satisfies condition (3), and $p(\cdot)$ satisfies condition (6). Then

$$\|\chi_{B(x,r)}\|_{p(\cdot)}\leqslant C[\mu(B(x,r))]^{\frac{1}{p(x)}}$$

with $C>0$ independent of $x\in X$ and $r>0$.

Proof of Theorem 1.3 It suffices to prove that T is bounded on $\dot K_{q(\cdot)}^{\alpha,p}(X)$. Suppose of $f\in\dot K_{q(\cdot)}^{\alpha,p}(X)$. By Theorem 1.1, $f(x)=\sum_{j\in\mathbb{Z}}\lambda_j b_j(x)$, where each b_j is a central $(\alpha,q(\cdot))$-block with support contained in B_j and

$$\|f\|_{\dot K_{q(\cdot)}^{\alpha,p}(X)}\approx\Big(\sum_{j\in\mathbb{Z}}|\lambda_j|^p\Big)^{1/p}.$$

Therefore, we get

$$\|Tf\|^p_{\dot K_{q(\cdot)}^{\alpha,p}(X)}$$
$$=C\sum_{k\in\mathbb{Z}}\mu(B_k)^{\alpha p}\|(Tf)\chi_k\|^p_{L^{q(\cdot)}(X)}$$
$$\leqslant C\Big[\sum_{k\in\mathbb{Z}}\mu(B_k)^{\alpha p}\times\Big(\sum_{j=-\infty}^{k-2}|\lambda_j|\;\|(Tb_j)\chi_k\|_{L^{q(\cdot)}(X)}\Big)^p+\sum_{k\in\mathbb{Z}}\mu(B_k)^{\alpha p}\times\Big(\sum_{j=k-1}^{\infty}|\lambda_j|\;\|(Tb_j)\chi_k\|_{L^{q(\cdot)}(X)}\Big)^p\Big]$$
$$=:C(I_1+I_2).$$

Let us first estimate I_1. By (5) and (10) we get

$$|Tb_j(x)|$$
$$\leqslant C\mu(B(x_0,d(x_0,x)))^{-1}\int_{B_j}|b_j(y)|\,\mathrm{d}\mu(y)$$
$$\leqslant C\mu(B_k)^{-1}\|b_j\|_{L^{q(\cdot)}(X)}\|\chi_{B_j}\|_{L^{q'(\cdot)}(X)}$$
$$\leqslant C\mu(B_k)^{-1}\mu(B_j)^{-\alpha}\|\chi_{B_j}\|_{L^{q'(\cdot)}(X)}.$$

So by Lemma 1.1 we have

$$\|(Tb_j)\chi_k\|_{L^{q(\cdot)}(X)}$$
$$\leqslant C\mu(B_k)^{-1}\mu(B_j)^{-\alpha}\|\chi_{B_j}\|_{L^{q'(\cdot)}(X)}\times\|\chi_{B_k}\|_{L^{q(\cdot)}(X)}$$
$$\leqslant C\mu(B_k)^{-1}\mu(B_j)^{-\alpha}\mu(B_j)^{1-\frac{1}{q(x)}}\mu(B_k)^{\frac{1}{q(x)}}$$
$$=C\mu(B_k)^{-1+\frac{1}{q(x)}}\mu(B_j)^{-\alpha+1-\frac{1}{q(x)}}.\tag{11}$$

Therefore, when $0<p\leqslant 1$, by $0<\alpha<1-\frac{1}{q^-}$, we get

$$I_1=\sum_{k\in\mathbb{Z}}\mu(B_k)^{\alpha p}\times$$

$$\left(\sum_{j=-\infty}^{k-2}|\lambda_j|\ \|(Tb_j)\chi_k\|_{L^{q(\cdot)}(X)}\right)^p$$

$$\leqslant C\sum_{k\in\mathbb{Z}}\mu(B_k)^{\alpha p}\Big(\sum_{j=-\infty}^{k-2}|\lambda_j|^p\times$$

$$\mu(B_k)^{[-1+\frac{1}{q(x)}]p}\mu(B_j)^{[-\alpha+1-\frac{1}{q(x)}]p}\Big)$$

$$\leqslant C\sum_{j\in\mathbb{Z}}|\lambda_j|^p\sum_{k\geqslant j+2}\left(\frac{\mu(B_j)}{\mu(B_k)}\right)^{[-\alpha+1-\frac{1}{q(x)}]p}$$

$$\leqslant C\sum_{j\in\mathbb{Z}}|\lambda_j|^p\sum_{k\geqslant j+2}A_0^{(j-k)[-\alpha+1-\frac{1}{q(x)}]p}$$

$$\leqslant C\sum_{j\in\mathbb{Z}}|\lambda_j|^p\leqslant C\|f\|^p_{\dot K^{\alpha,p}_{q(\cdot)}(X)}. \tag{12}$$

When $1<p<\infty$, take $1/p+1/p'=1$. Since $0<\alpha<1-\frac{1}{q^-}$, by (11) and the Hölder inequality, we have

$$I_1\leqslant C\sum_{k\in\mathbb{Z}}\mu(B_k)^{\alpha p}\Big(\sum_{j=-\infty}^{k-2}|\lambda_j|\ \mu(B_k)^{-1+\frac{1}{q(x)}}\times$$

$$\mu(B_j)^{-\alpha+1-\frac{1}{q(x)}}\Big)^p$$

$$\leqslant C\sum_{k\in\mathbb{Z}}\left(\sum_{j=-\infty}^{k-2}|\lambda_j|^p\times\left(\frac{\mu(B_j)}{\mu(B_k)}\right)^{[-\alpha+1-\frac{1}{q(x)}]p/2}\right)\times$$

$$\left(\sum_{j=-\infty}^{k-2}\left(\frac{\mu(B_j)}{\mu(B_k)}\right)^{[-\alpha+1-\frac{1}{q(x)}]p'/2}\right)^{p/p'}$$

$$\leqslant C\sum_{k\in\mathbb{Z}}\Big(\sum_{j=-\infty}^{k-2}|\lambda_j|^pA_0^{(j-k)[-\alpha+1-\frac{1}{q(x)}]p/2}\Big)$$

$$=C\sum_{j\in\mathbb{Z}}|\lambda_j|^p\sum_{k\geqslant j+2}A_0^{(j-k)[-\alpha+1-\frac{1}{q(x)}]p/2}$$

$$\leqslant C\sum_{j\in\mathbb{Z}}|\lambda_j|^p\leqslant C\|f\|^p_{\dot K^{\alpha,p}_{q(\cdot)}(X)}. \tag{13}$$

Let us now estimate I_2. Similarly, we consider two cases for p. When $0<p\leqslant1$, by $L^{q(\cdot)}(X)$ boundedness of T, we have

$$I_2=\sum_{k\in\mathbb{Z}}\mu(B_k)^{\alpha p}\times$$

$$\Big(\sum_{j=k-1}^{\infty}|\lambda_j|\,\|(Tb_j)\chi_k\|_{L^{q(\cdot)}(X)}\Big)^p$$

$$\leqslant C\sum_{k\in\mathbb{Z}}\mu(B_k)^{\alpha p}\Big(\sum_{j=k-1}^{\infty}|\lambda_j|^p\,\|b_j\|^p_{L^{q(\cdot)}(X)}\Big)$$

$$\leqslant C\sum_{k\in\mathbb{Z}}\mu(B_k)^{\alpha p}\Big(\sum_{j=k-1}^{\infty}|\lambda_j|^p\mu(B_j)^{-\alpha p}\Big)$$

$$=C\sum_{j\in\mathbb{Z}}|\lambda_j|^p\sum_{k\leqslant j+1}\left(\frac{\mu(B_k)}{\mu(B_j)}\right)^{\alpha p}$$

$$\leqslant C\sum_{j\in\mathbb{Z}}|\lambda_j|^p\leqslant C\|f\|^p_{\dot K^{\alpha,p}_{q(\cdot)}(X)}. \tag{14}$$

When $1<p<\infty$. take $1/p+1/p'=1$. By $L^{q(\cdot)}(X)$ boundedness of T and the Hölder inequality. we have

$$I_2\leqslant C\sum_{k\in\mathbb{Z}}\mu(B_k)^{\alpha p}\times\Big(\sum_{j=k-1}^{\infty}|\lambda_j|^p\,\|(Tb_j)\chi_k\|^{p/2}_{L^{q(\cdot)}(X)}\Big)\times$$

$$\Big(\sum_{j=k-1}^{\infty}\|(Tb_j)\chi_k\|^{p'/2}_{L^{q(\cdot)}(X)}\Big)^{p/p'}$$

$$\leqslant C\sum_{k\in\mathbb{Z}}\mu(B_k)^{\alpha p}\Big(\sum_{j=k-1}^{\infty}|\lambda_j|^p\,\|b_j\|^{p/2}_{L^{q(\cdot)}(X)}\Big)\times$$

$$\Big(\sum_{j=k-1}^{\infty}\|b_j\|^{p'/2}_{L^{q(\cdot)}(X)}\Big)^{p/p'}$$

$$\leqslant C\sum_{k\in\mathbb{Z}}\left(\sum_{j=k-1}^{\infty}|\lambda_j|^p\left(\frac{\mu(B_k)}{\mu(B_j)}\right)^{\alpha p/2}\right)\times$$

$$\left(\sum_{j=k-1}^{\infty}\left(\frac{\mu(B_k)}{\mu(B_j)}\right)^{\alpha p'/2}\right)^{p/p'}$$

$$\leqslant C\sum_{k\in\mathbb{Z}}\left(\sum_{j=k-1}^{\infty}|\lambda_j|^p\left(\frac{\mu(B_k)}{\mu(B_j)}\right)^{\alpha p/2}\right)$$

$$=C\sum_{j\in\mathbb{Z}}|\lambda_j|^p\sum_{k\leqslant j+1}\left(\frac{\mu(B_k)}{\mu(B_j)}\right)^{\alpha p/2}$$

$$\leqslant C\sum_{j\in\mathbb{Z}}|\lambda_j|^p\leqslant C\|f\|^p_{\dot K^{\alpha,p}_{q(\cdot)}(X)}. \tag{15}$$

Combining inequalities (12) - (15), we have

$$\|Tf\|_{\dot K^{\alpha,p}_{q(\cdot)}(X)}\leqslant C\|f\|_{\dot K^{\alpha,p}_{q(\cdot)}(X)}.$$

Thus, the proof of Theorem 1.3 is completed.

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