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A note of Riesz transform on $\mathscr{D}(\mathbb{R}^n)$ *

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Abstract　In this work, we prove that, for a non-zero function $f \in \mathscr{D}(\mathbb{R}^n)$, its Riesz transform $R_j f$ does not have compact support, which improves the known result of Hilbert transform.

Keywords　Riesz transform; $\mathscr{D}(\mathbb{R}^n)$; compact support

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$\mathscr{D}(\mathbb{R}^n)$ 函数的 Riesz 变换的一个注记

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摘　要　证明,对于任意一个非零 $\mathscr{D}(\mathbb{R}^n)$ 函数 f, 它的 Riesz 变换 $R_j f$ 不具有紧支集。这推广了已知的 Hilbert 变换的结果。

关键词　Riesz 变换; $\mathscr{D}(\mathbb{R}^n)$; 紧支集

For $f \in L^p(\mathbb{R})$, $1 \leqslant p < \infty$, the classical Hilbert transform is defined as

$$Hf(x) := \frac{1}{\pi}\mathrm{p.v.}\int_{\mathbb{R}} \frac{f(t)}{x-t}\mathrm{d}t, \tag{1}$$

where "p. v." is the Cauchy principal value (see Ref. [1]), that is

$$\mathrm{p.v.}\int_{\mathbb{R}} \frac{f(t)}{x-t}\mathrm{d}t = \lim_{\varepsilon\to 0}\int_{|x-t|>\varepsilon} \frac{f(t)}{x-t}\mathrm{d}t.$$

Yang[2] proved that $\mathscr{D}(\mathbb{R}) \cap H(\mathscr{D}(\mathbb{R})) = \{0\}$, where

$$\mathscr{D}(\mathbb{R}^n) = \{\phi : \phi \in C_c^\infty(\mathbb{R}^n),$$
$$\forall \alpha \in N_0^n,\ \rho_\alpha(\phi) = \sup_{x\in\mathbb{R}^n} |D^\alpha \phi(x)| < \infty\},$$

for $n \in N$.

That is to say the function Hf does not have compact support for all non-zero function $f \in \mathscr{D}(\mathbb{R})$.

Consider n-dimensional Euclidean space $\mathbb{R}^n$ ($n \in \mathrm{N}$). For $f \in L^p(\mathbb{R}^n)$, $1 \leqslant p < \infty$, $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$, $t = (t_1, t_2, \cdots, t_n) \in \mathbb{R}^n$, the n-dimensional Hilbert transform H_n is defined as

$$H_n f(x) := \frac{1}{\pi^n}\mathrm{p.v.}\int_{\mathbb{R}^n} \frac{f(t)}{(x_1-t_1)(x_2-t_2)\cdots(x_n-t_n)}\mathrm{d}t. \tag{2}$$

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For $n = 2$ Cui et al.[3] obtained the result:

$$\mathscr{D}(\mathbb{R}^2) \cap H_2(\mathscr{D}(\mathbb{R}^2)) = \{0\}.$$

When $n > 2$, Shen[4] proved

$$\mathscr{D}(\mathbb{R}^n) \cap H_n(\mathscr{D}(\mathbb{R}^n)) = \{0\}. \tag{3}$$

We have known that Hilbert transform has a relation with Fourier transform which can be presented as

$$\widehat{Hf}(\xi) = -i \cdot \mathrm{sgn}(\xi)\hat{f}(\xi), \tag{4}$$

where

$$\mathrm{sgn}(x) = \begin{cases} 1 & \text{if} \quad x > 1, \\ 0 & \text{if} \quad n = 0, \\ -1 & \text{if} \quad x < 0. \end{cases}$$

For $f \in L^p(\mathbb{R}^n)$, $1 \leqslant p < \infty$, $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$, the Riesz transform R_j ($j \in \{1, 2, \cdots, n\}$) is defined as

$$R_j f(x) := \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \mathrm{p.\,v.} \int_{\mathbb{R}^n} \frac{x_j - y_j}{|x-y|^{n+1}} f(y)\,\mathrm{d}y,$$

and we have (see Ref. [1])

$$\widehat{R_j f}(\xi) = -i \cdot \frac{\xi_j}{|\xi|}\hat{f}(\xi). \tag{5}$$

If $n = 1$, the identity (5) will reduce to (4). In this sense, Riesz transform is the higher-dimensional form of Hilbert transform. Motivated by Refs. [2-4], we shall consider whether the equation

$$\mathscr{D}(\mathbb{R}^n) \cap R_j(\mathscr{D}(\mathbb{R}^n)) = \{0\} \tag{6}$$

holds for $n > 1$.

In section 1, we will give an affirmative answer for (6).

1 Main results

Before we prove our main theorem, we need the following lemmas.

Lemma 1.1 Suppose that $f \in L^1(\mathbb{R}^n)$ with compact support. Then we have $\hat{f} \in C^\infty(\mathbb{R}^n)$.

This lemma is a basic result which appears in many books on real analysis and we refer readers to Ref. [5].

Lemma 1.2 Suppose that $f, g \in C^\infty(\mathbb{R}^n)$ and $g(x) = \frac{x_j}{|x|}f(x)$ for some $j \in \{1, \cdots, n\}$. Then we have $\left(\frac{\partial}{\partial x}\right)^\alpha f(0) = 0$ for all multi-index $\alpha \in \mathbb{N}_0^n$.

Proof Considering the continuity of the function g, we have

$$g(0) = \lim_{x \to 0} \frac{x_j}{|x|} f(0). \tag{7}$$

The equality (7) implies that $f(0) = 0$.

When $|\alpha| = 1$, taking derivatives on g, we have

$$\frac{\partial g}{\partial x_i}(x) = \frac{\partial f}{\partial x_i}(x)\frac{x_j}{|x|} + f(x)\frac{\partial}{\partial x_i}\left(\frac{x_j}{|x|}\right), \ i \neq j;$$

$$\frac{\partial g}{\partial x_j}(x) = \frac{\partial f}{\partial x_j}(x)\frac{x_j}{|x|} + f(x)\frac{\partial}{\partial x_j}\left(\frac{x_j}{|x|}\right). \tag{8}$$

By substituting $x = 0$ into (8), there hold

$$\frac{\partial g}{\partial x_i}(0) = \frac{\partial f}{\partial x_i}(0)\lim_{x \to 0}\frac{x_j}{|x|}, \ i \neq j;$$

$$\frac{\partial g}{\partial x_j}(0) = \frac{\partial f}{\partial x_j}(0)\lim_{x \to 0}\frac{x_j}{|x|}. \tag{9}$$

Since $\frac{\partial g}{\partial x_i}(0)$ and $\frac{\partial g}{\partial x_j}(0)$ all exist and $\lim \frac{x_i}{|x|}$, $(i = 1, \cdots, n)$ does not exist while x tends to 0, we derive from (9) that

$$\frac{\partial f}{\partial x_i}(0) = 0, \tag{10}$$

for all $1 \leqslant i \leqslant n$.

Now we use induction method to solve the case $|\alpha| \geqslant 2$.

Suppose that for any multiple-index β with $|\beta| < |\alpha|$, there holds

$$\left(\frac{\partial}{\partial x}\right)^\beta f(0) = 0. \tag{11}$$

Furthermore,

$$\left(\frac{\partial}{\partial x}\right)^\alpha g(x) = \frac{x_j}{|x|}\left(\frac{\partial}{\partial x}\right)^\alpha f(x) + \sum_{\gamma+\delta=\alpha}\left(\frac{\partial}{\partial x}\right)^\gamma f(x)\left(\frac{\partial}{\partial x}\right)^\delta\left(\frac{x_j}{|x|}\right). \tag{12}$$

The assumption $g \in C^\infty(\mathbb{R}^n)$ implies that $\left(\frac{\partial}{\partial x}\right)^\alpha g(x)$ exists for all $x \in \mathbb{R}^n$. Then we obtain from (11) and (12) that

$$\left(\frac{\partial}{\partial x}\right)^\alpha g(0) = \lim_{x \to 0}\frac{x_j}{|x|}\left(\frac{\partial}{\partial x}\right)^\alpha f(0). \tag{13}$$

Thus we immediately get $\left(\frac{\partial}{\partial x}\right)^\alpha f(0) = 0$ from (13). □

Now we shall prove our main theorem.

Theorem 1.1 Suppose that $f \in \mathscr{D}(\mathbb{R}^n)$. If $R_j f$ has compact support, then $\widehat{R_j f} \in C^\infty(\mathbb{R}^n)$ and $f \equiv 0$.

Proof First we obtain $f \in L^2(\mathbb{R}^n)$ from $f \in \mathscr{D}(\mathbb{R}^n)$.

Since the Riesz transform R_j is of strong type (p, p) with $1 < p < \infty$, we obtain that $R_j f$ is also in $L^2(\mathbb{R}^n)$. Furthermore, $R_j f \subset L^1(\mathbb{R}^n)$ from the assumption that $R_j f$ has compact support. By Lemma 1.1, there holds $\widehat{R_j f} \subset C^\infty(\mathbb{R}^n)$.

Since $f \in \mathscr{D}(\mathbb{R}^n)$ means that $f \in L^1(\mathbb{R}^n)$ and f has compact support, by using Lemma 1.1 again, it yields $\hat{f} \in C^\infty(\mathbb{R}^n)$.

Noting $\widehat{R_j f}(\xi) = -i \dfrac{\xi_j}{|\xi|} \hat{f}(\xi)$ by (5), we conclude

$$\left(\frac{\partial}{\partial x}\right)^\alpha \hat{f}(0) = 0, \tag{14}$$

for all $\alpha \in \mathbb{N}_0^n$ by Lemma 1.2.

That is to say,

$$\int_{\mathbb{R}^n} (-2\pi i x)^\alpha f(x)\,\mathrm{d}x = 0 \tag{15}$$

holds for all $\alpha \in \mathbb{N}_0^n$.

Let $\operatorname{supp} f \in Q$, then the identity

$$\int_Q P(x) f(x)\,\mathrm{d}x = 0 \tag{16}$$

holds for all polynomials P by (15).

The collection of all continuous functions defined on Q is denoted by $C(Q)$. Considering that the polynomials are dense in the functional space $C(Q)$, it immediately follows $f \equiv 0$ from (16). □

Obviously, Theorem 1.1 implies (6), which is our main conclusion.

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