文章编号:2095-6134(2018)02-0145-09

Hadamard-type inequalities for products of (h, m)-convex functions and their applications*

SUN Wenbing†

(*School of Science*, *Shaoyang University*, *Shaoyang* 422000, *Hunan*, *China*)

(Received 30 December 2016; Revised 13 April 2017)

Hadamard-type inequalities for products of (h, m)-convex functions and their applications[J]. Journal of University of Chinese Academy of Sciences, 2018,35(2):145-153.

Abstract In this paper, we established some new Hadamard-type inequalities for products of (h, m)-convex functions, which are the extended Hadamard-type inequalities for ordinary convexity sense, s-convexity in the second sense, m-convexity sense, and h-convexity sense.

Keywords Hadamard's inequality; convex functions; (h, m)-convex functions; product of two functions

CLC number:O178 **Document code**: A **doi**:10.7523/j.issn.2095-6134.2018.02.001

关于(h, m)-凸函数乘积的 Hadamard-型不等式及应用

孙文兵

(邵阳学院理学院,湖南 邵阳 422000)

摘 要 建立一些关于(h, m)-凸函数乘积的新 Hadamard-型不等式,得到的结果是对通常凸性、第 2 种意义下的 s-凸性、m-凸性、h-凸性意义下的 Hadamard-型不等式的推广.

关键词 Hadamard 不等式;凸函数;(h, m)-凸函数;2 个函数乘积

In recent years, the concept of convex function has been extended by some scholars. For example, Breckner[1] introduced the concept of s-convexity, and Varošanec[2] defined h-convex functions. Some results for Hadamard-type inequalities related to the extended convex functions have been obtained[3-7].

1 Background knowledge

In 2011, Özdemir et al.[8] presented the (h, m)-convex function as follows.

Definition 1.1 Let $h: J \subseteq \mathbb{R} \to \mathbb{R}$ be a non-negative function. We say that $f: [0, b] \to \mathbb{R}$ is an (h, m)-convex function with $m \in [0, 1]$, if f is non-negative and for all $x, y \in [0, b]$ and $\alpha \in (0, 1)$, we have

$$f(\alpha x + m(1-\alpha)y) \leqslant h(\alpha)f(x) + mh(1-\alpha)f(y).$$

If the above inequality is reversed, f is said to be (h, m)-concave function on $[0, b]$.

* Supported by National Natural Science Foundation of China (61672356) and Natural Science Foundation of Hunan Province(12JJ3008)

† E-mail: swb0520@163.com

Remark 1.1

1) if we choose $m = 1$, we have h-convex functions;

2) if we choose $m=1$ and $h(\alpha)=\alpha$, we obtain non-negative ordinary convex functions;

3) if we choose $m=1$ and $h(\alpha)=\alpha^s$, we have s-convex functions in the second sense;

4) if we choose $h(\alpha)=\alpha$, we have m-convex functions.

One of important applications of the concept of convex function is the Hadamard's inequality as follows.

Let $f: I\subseteq \mathbb{R} \to \mathbb{R}$ be a convex function and $a, b \in I$ with $a < b$, then the inequality

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a} \leqslant \int_a^b f(x)\,\mathrm{d}x \leqslant \frac{f(a)+f(b)}{2} \tag{1}$$

holds, which is well known as Hadamard's inequality.

In Ref. [9], Dragomir and Fitzpatrick established the Hadamard's type inequalities for s-convex function as follows.

Theorem 1.1 Suppose that $f: [0, \infty) \to [0, \infty)$ is an s-convex function in the second sense, where $s\in(0, 1]$, and let $a,b\in[0, \infty)$, $a<b$. If $f\in L^1([a, b])$, the inequalities

$$2^{s-1}f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \leqslant \frac{f(a)+f(b)}{s+1} \tag{2}$$

hold. The constant $k=\dfrac{1}{s+1}$ is the best possible in the second inequality in (2).

In Ref. [3], Sarkaya proved the Hadamard's type inequalities for class of h-convex functions as follows.

Theorem 1.2 Let $f: I\subseteq\mathbb{R}\to\mathbb{R}$ be an h-convex function, $a,b\in I$, $a<b$ and $f\in L^1[a,b]$. Then

$$\frac{1}{2h\left(\frac{1}{2}\right)}f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \leqslant [f(a)+f(b)]\int_0^1 h(t)\,\mathrm{d}t. \tag{3}$$

In Ref. [5], Dragomir and Toader proved the inequality for m-convex function as follows.

Theorem 1.3 Let $f: [0, \infty) \to \mathbb{R}$ be an m-convex function with $m\in(0,1]$. If $0\leqslant a<b<\infty$ and $f\in L^1[a,b]$, one has the inequality

$$\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}t \leqslant \min\left\{\frac{f(a)+mf\left(\frac{b}{m}\right)}{2}, \frac{f(b)+mf\left(\frac{a}{m}\right)}{2}\right\}. \tag{4}$$

Some inequalities of Hadamard-type related to this new class of (h, m)-convex functions are given[8].

Theorem 1.4 Let $f: [0, \infty) \to \mathbb{R}$ be an (h, m)-convex function with $m\in(0,1]$ and $t\in[0,1]$. If $0\leqslant a<b<\infty$ and $f\in L^1[a,b]$, the inequality

$$\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}t \leqslant \min\left\{f(a)\int_0^1 h(t)\,\mathrm{d}t + mf\left(\frac{b}{m}\right)\int_0^1 h(1-t)\,\mathrm{d}t,\ f(b)\int_0^1 h(t)\,\mathrm{d}t + mf\left(\frac{a}{m}\right)\int_0^1 h(1-t)\,\mathrm{d}t\right\} \tag{5}$$

holds.

In Ref. [10], Pachpatte established some Hadamard's type inequalities for products of convex functions as follows.

Theorem 1.5 Let $f, g: [a,b] \to [0,\infty)$ be convex functions on $[a, b]\in\mathbb{R}$, $a<b$, then

$$\frac{1}{b-a}\int_a^b f(x)g(x)\,\mathrm{d}x \leqslant \frac{1}{3}M(a,b) + \frac{1}{6}N(a,b), \tag{6}$$

and

$$2f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)g(x)\,\mathrm{d}x + \frac{1}{6}M(a,b) + \frac{1}{3}N(a,b), \tag{7}$$

where $M(a, b) = f(a)g(a) + f(b)g(b)$, $N(a,b) = f(a)g(b) + f(b)g(a)$.

In Ref. [11], some Hadamard's type inequalities for products of s-convex functions in the second sense are constructed by Kirmaci et al. as follows.

Theorem 1.6 Let $f,g: [a,b]\to\mathbb{R}$, $a,b\in[0, \infty)$, $a<b$, be functions such that f and fg are in $L^1([a,b])$. If f is convex and nonnegative on $[a,b]$ and if g is s-convex on $[a,b]$ for some fixed $s\in(0,1)$, then

$$\frac{1}{b-a}\int_a^b f(x)g(x)\,dx \leqslant$$

$$\frac{1}{s+2}M(a,b)+\frac{1}{(s+1)(s+2)}N(a,b), \quad (8)$$

where $M(a,b)=f(a)g(a)+f(b)g(b)$, $N(a,b)=f(a)g(b)+f(b)g(a)$.

Theorem 1.7 Let $f,g:[a,b]\to\mathbb{R}$, $a,b\in[0,\infty)$, $a<b$, be functions such that f and fg in $L^1([a,b])$. If f is s_1-convex and g is s_2-convex connegative on $[a,b]$ for some fixed $s_1,s_2\in(0,1)$, then

$$\frac{1}{b-a}\int_a^b f(x)g(x)\,dx \leqslant$$

$$\frac{1}{s_1+s_2+1}M(a,b)+B(s_1+1,s_2+1)N(a,b)=$$

$$\frac{1}{s_1+s_2+1}\left[M(a,b)+s_1s_2\frac{\Gamma(s_1)\Gamma(s_2)}{\Gamma(s_1+s_2+1)}N(a,b)\right], \quad (9)$$

where $B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}dt=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$.

Theorem 1.8 Let $f,g:[a,\mathrm{b}]\to\mathbb{R}$, $\mathrm{a},\mathrm{b}\in[0,\infty)$, $a<b$, be functions such that f and fg are in $L^1([a,b])$. If f is convex and nonnegative on $[a,b]$ and if g is s-convex on $[a,b]$ for some fixed $s\in(0,1)$, then

$$2^s f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)g(x)\,dx \leqslant$$

$$\frac{1}{(s+1)(s+2)}M(a,b)+\frac{1}{s+2}N(a,b). \quad (10)$$

The main purpose of this work is to establish some new Hadamard-type inequalities, similar to the above inequalities, for products of convex functions and $(h,\ m)$-convex functions, which are generalizations of the above inequalities.

2 Main results and applications

Theorem 2.1 Let $h:[0,1]\to(0,\infty)$, $f,\ g:[0,\infty)\to\mathbb{R}$, be functions such that $h\in L^1([0,1])$, $f\,g\in L^1([a,b])$ with $a,b\in[0,\infty)$, $a<b$. If f is convex and nonnegative on $[0,\infty)$, and if g is (h,m)-convex and nonnegative on $[0,\infty)$ with $m\in(0,1]$ and $t\in[0,1]$, then the following inequality holds,

$$\frac{1}{b-a}\int_a^b f(x)g(x)\,dx$$

$$\leqslant \min\{[f(a)-f(b)]P(t,h(t),h(1-t))+ f(b)P(1,h(t),h(1-t)),\ [f(b)-f(a)]Q(t,h(t),h(1-t))+ f(a)Q(1,h(t),h(1-t))\}, \quad (11)$$

where

$$P(t,h(t),h(1-t)) = g(a)\int_0^1 th(t)\,dt+mg\left(\frac{b}{m}\right)\int_0^1 th(1-t)\,dt,$$

$$P(1,h(t),h(1-t)) = g(a)\int_0^1 h(t)\,dt+mg\left(\frac{b}{m}\right)\int_0^1 h(1-t)\,dt,$$

$$Q(t,h(t),h(1-t)) = g(b)\int_0^1 th(t)\,dt+mg\left(\frac{a}{m}\right)\int_0^1 th(1-t)\,dt,$$

$$Q(1,h(t),h(1-t)) = g(b)\int_0^1 h(t)\,dt+mg\left(\frac{a}{m}\right)\int_0^1 h(1-t)\,dt.$$

Proof Since f is convex and nonnegative on $[0,\infty)$, we have

$$f(ta+(1-t)b)\leqslant tf(a)+(1-t)f(b)$$

and

$$f(tb+(1-t)a)\leqslant tf(b)+(1-t)f(a).$$

From g is $(h,\ m)$-convex and nonnegative on $[0,\infty)$, that is

$$g(tx+m(1-t)y)\leqslant h(t)g(x)+mh(1-t)g(y),$$

for all $x,y\in[0,\infty)$. It follows that, for all $t\in[0,1]$,

$$g(ta+(1-t)b)\leqslant h(t)g(a)+mh(1-t)g\left(\frac{b}{m}\right),$$

and

$$g(tb+(1-t)a)\leqslant h(t)g(b)+mh(1-t)g\left(\frac{a}{m}\right).$$

By the nonnegativeness of f and g, we obtain

$$f(ta+(1-t)b)g(ta+(1-t)b)$$

$$\leqslant [tf(a)+(1-t)f(b)]\left[h(t)g(a)+mh(1-t)g\left(\frac{b}{m}\right)\right]$$

$$=[f(a)-f(b)]\left[g(a)th(t)+mg\left(\frac{b}{m}\right)th(1-t)\right]+f(b)\left[g(a)h(t)+mg\left(\frac{b}{m}\right)h(1-t)\right],$$

and

$$f(tb+(1-t)b)g(tb+(1-t)a)$$

$$\leqslant [tf(b)+(1-t)f(a)][h(t)g(b)+$$

$$mh(1-t)g(\frac{a}{m})]$$
$$=[f(b)-f(a)][g(b)th(t)+$$
$$mg(\frac{a}{m})th(1-t)]+$$
$$f(a)[g(b)h(t)+mg(\frac{a}{m})h(1-t)].$$

Integrating the above two inequalities on $[0,1]$, with respect to t, we obtain

$$\int_0^1 f(ta+(1-t)b)g(ta+(1-t)b)\mathrm{d}t$$
$$\leqslant [f(a)-f(b)][g(a)\int_0^1 th(t)\mathrm{d}t+$$
$$mg(\frac{b}{m})\int_0^1 th(1-t)\mathrm{d}t]+$$
$$f(b)[g(a)\int_0^1 (h(t)\mathrm{d}t+mg(\frac{b}{m})\int_0^1 h(1-t)\mathrm{d}t)],$$

and

$$\int_0^1 f(tb+(1-t)a)g(tb+(1-t)a)\mathrm{d}t$$
$$\leqslant [f(b)-f(a)][g(b)\int_0^1 th(t)\mathrm{d}t+$$
$$mg(\frac{b}{m})\int_0^1 th(1-t)\mathrm{d}t]+$$
$$f(a)[g(b)\int_0^1 (h(t)\mathrm{d}t+mg(\frac{b}{m})\int_0^1 h(1-t)\mathrm{d}t)].$$

It is easy to see

$$\int_0^1 f(ta+(1-t)b)g(ta+(1-t)b)\mathrm{d}t$$
$$=\int_0^1 f(tb+(1-t)a)g(tb+(1-t)a)\mathrm{d}t$$
$$=\frac{1}{b-a}\int_a^b f(x)g(x)\mathrm{d}x.$$

Using the above inequalities and equality, we obtain the required result.

Remark 2.1 If we choose $f(x)=1$ in (11) for all $x\in[a,b]$, we obtain the inequality (5).

Remark 2.2 If we choose $f(x)=1$ and $h(t)=t$ in (11) for $x\in[a,b]$, we obtain the inequality (4).

Remark 2.3 If we choose $f(x)=1, m=1$ and $h(t)=t$ in (11), we obtain the right hand side of the Hadamard's inequality (1). If we choose $f(x)=1, m=1$ and $h(t)=t^s$ in (11), we obtain the right hand side of the inequality (2).

Theorem 2.2 Let $h_1, h_2:[0,1]\rightarrow(0,\infty), f, g:[0,\infty)\rightarrow\mathbb{R}$, be functions such that $h_1h_2\in L^1([0,1])$, $fg\in L^1([a,b])$ with $a,b\in[0,\infty)$, $a<b$. If f is (h_1, m_1)-convex and nonnegative on $[0,\infty)$, and if g is (h_2, m_2)-convex and nonnegative on $[0,\infty)$ with $m_1, m_2\in(0,1]$ and $t\in[0,1]$, then the following inequality holds,

$$\frac{1}{b-a}\int_a^b f(x)g(x)\mathrm{d}x$$
$$\leqslant \min\{f(a)P(h_1(t),h_2(t),h_2(1-t))+$$
$$m_1f(\frac{b}{m_1})P(h_1(1-t),h_2(t),h_2(1-t)),$$
$$f(b)Q(h_1(t),h_2(t),h_2(1-t))+$$
$$m_1f(\frac{a}{m_1})Q(h_1(1-t),h_2(t),h_2(1-t))\},\tag{12}$$

where

$$P(h_1(t),h_2(t),h_2(1-t))$$
$$=g(a)\int_0^1 h_1(t)h_2(t)\mathrm{d}t+$$
$$m_2g(\frac{b}{m_2})\int_0^1 h_1(t)h_2(1-t)\mathrm{d}t,$$
$$P(h_1(1-t),h_2(t),h_2(1-t))$$
$$=g(a)\int_0^1 h_1(1-t)h_2(t)\mathrm{d}t+$$
$$m_2g(\frac{b}{m_2})\int_0^1 h_1(1-t)h_2(1-t)\mathrm{d}t,$$
$$Q(h_1(t),h_2(t),h_2(1-t))$$
$$=g(b)\int_0^1 h_1(t)h_2(t)\mathrm{d}t+$$
$$m_2g(\frac{a}{m_2})\int_0^1 h_1(t)h_2(1-t)\mathrm{d}t,$$
$$Q(h_1(1-t),h_2(t),h_2(1-t))$$
$$=g(b)\int_0^1 h_1(1-t)h_2(t)\mathrm{d}t+$$
$$m_2g(\frac{a}{m_2})\int_0^1 h_1(1-t)h_2(1-t)\mathrm{d}t.$$

Proof Since f is (h_1, m_1)-convex and nonnegative on $[0,\infty)$, that is

$$f(tx+m_1(1-t)y)\leqslant h_1(t)f(x)+$$
$$m_1h_1(1-t)f(y)$$

for all $x,y\in[0,\infty)$, we have

$$f(ta+(1-t)b)\leqslant h_1(t)f(a)+m_1h_1(1-t)f(\frac{b}{m_1}),$$

and

$$f(tb+(1-t)a)\leqslant h_1(t)f(b)+m_1h_1(1-t)f(\frac{a}{m_1}).$$

From g is (h_2,m_2)-convex and nonnegative on $[0,$

∞), that is

$$g(tx+m_2(1-t)y)\leqslant h_2(t)g(x)+m_2h_2(1-t)g(y),$$

for all $x,y\in[0,\infty)$. It follows that, for all $t\in[0,1]$,

$$g(ta+(1-t)b)\leqslant h_2(t)g(a)+m_2h_2(1-t)g(\frac{b}{m_2})$$

and

$$g(tb+(1-t)a)\leqslant h_2(t)g(b)+m_2h_2(1-t)g(\frac{a}{m_2}).$$

By the nonnegativeness of f and g, we obtain

$$\begin{aligned}&f(ta+(1-t)b)g(ta+(1-t)b)\\&\leqslant[h_1(t)f(a)+m_1h_1(1-t)f(\frac{b}{m_1})]\times\\&[h_2(t)g(a)+m_2h_2(1-t)g(\frac{b}{m_2})]\\&=f(a)[g(a)h_1(t)h_2(t)+\\&m_2g(\frac{b}{m_2})h_1(t)h_2(1-t)]+\\&m_1f(\frac{b}{m_1})[g(a)h_1(1-t)h_2(t)+\\&m_2g(\frac{b}{m_2})h_1(1-t)h_2(1-t)],\end{aligned}$$

and

$$\begin{aligned}&f(tb+(1-t)a)g(tb+(1-t)a)\\&\leqslant[h_1(t)f(b)+m_1h_1(1-t)f(\frac{a}{m_1})]\times\\&[h_2(t)g(b)+m_2h_2(1-t)g(\frac{a}{m_2})]\\&=f(b)[g(b)h_1(t)h_2(t)+\\&m_2g(\frac{a}{m_2})h_1(t)h_2(1-t)]+\\&m_1f(\frac{a}{m_1})[g(b)h_1(1-t)h_2(t)+\\&m_2g(\frac{a}{m_2})h_1(1-t)h_2(1-t)].\end{aligned}$$

Integrating the above two inequalities on $[0,1]$, with respect to t, we obtain

$$\begin{aligned}&\int_0^1 f(ta+(1-t)b)g(ta+(1-t)b)\,dt\\&\leqslant f(a)[g(a)\int_0^1 h_1(t)h_2(t)\,dt]+\\&m_2g(\frac{b}{m_2})\int_0^1 h_1(t)h_2(1-t)\,dt+\\&m_1f(\frac{b}{m_1})[g(a)\int_0^1 h_1(1-t)h_2(t)\,dt]+\\&m_2g(\frac{b}{m_2})\int_0^1 h_1(1-t)h_2(1-t)\,dt],\end{aligned}$$

and

$$\begin{aligned}&\int_0^1 f(tb+(1-t)a)g(tb+(1-t)a)\,dt\\&\leqslant f(b)[g(a)\int_0^1 h_1(t)h_2(t)\,dt]+\\&m_2g(\frac{a}{m_2})\int_0^1 h_1(t)h_2(1-t)\,dt+\\&m_1f(\frac{a}{m_1})[g(b)\int_0^1 h_1(1-t)h_2(t)\,dt]+\\&m_2g(\frac{a}{m_2})\int_0^1 h_1(1-t)h_2(1-t)\,dt],\end{aligned}$$

It is easy to see that

$$\begin{aligned}&\int_0^1 f(ta+(1-t)b)g(ta+(1-t)b)\,dt\\&=\int_0^1 f(tb+(1-t)a)g(tb+(1-t)a)\,dt\\&=\frac{1}{b-a}\int_a^b f(x)g(x)\,dx.\end{aligned}$$

Using the above three inequalities and equality, we obtain the required result.

Remark 2.4 If we choose $h_1(t)=t, m_1=1$ in (12), we can obtain the inequality (11).

Corollary 2.1 Let $h:[0,1]\to(0,\infty)$, f, g: $[0,\infty)\to\mathbb{R}$, be functions such that $h\in L^1([0,1])$, $fg\in L^1([a,b])$ with $a,b\in[0,\infty)$, $a<b$. If f and g is $(h,\ m)$-convex and nonnegative on $[0,\infty)$ with $m\in(0,1]$ and $t\in[0,1]$, then the following inequality holds,

$$\begin{aligned}&\frac{1}{b-a}\int_a^b f(x)g(x)\,dx\\&\leqslant\min\{f(a)P(h(t),h(t),h(1-t))+\\&mf(\frac{b}{m})P(h(1-t),h(t),h(1-t)),\\&f(b)Q(h(t),h(t),h(1-t))+\\&mf(\frac{a}{m})Q(h(1-t),h(t),h(1-t))\},\end{aligned}\tag{13}$$

where

$$\begin{aligned}&P(h(t),h(t),h(1-t))\\&=g(a)\int_0^1 h^2(t)\,dt+mg(\frac{b}{m})\int_0^1 h(t)h(1-t)\,dt,\\&P(h(1-t),h(t),h(1-t))\\&=g(a)\int_0^1 h(1-t)h(t)\,dt+mg(\frac{b}{m})\int_0^1 h(1-t)^2\,dt,\\&Q(h(t),h(t),h(1-t))\\&=g(b)\int_0^1 h^2(t)\,dt+mg(\frac{a}{m})\int_0^1 h(t)h(1-t)\,dt,\\&Q(h(1-t),h(t),h(1-t))\end{aligned}$$

$$= g(b)\int_0^1 h(1-t)h(t)\,\mathrm{d}t + mg\left(\frac{a}{m}\right)\int_0^1 h^2(1-t)\,\mathrm{d}t.$$

Proof From Theorem 2.2 let $h_1 = h_2 = h$ and $m_1 = m_2 = m$, so Corollary 2.1 immediately holds.

Theorem 2.3 Let $h:[0,1]\to(0,\infty)$, f, g: $[0,\infty)\to\mathbb{R}$, be functions such that $h\in L^1([0,1])$, $fg\in L^1([a,b])$ with $a,b\in[0,\infty)$, $a<b$. If f is convex and nonnegative on $[0,\infty)$, and g is (h,m)-convex and nonnegative on $[0,\infty)$ with $m\in(0,1]$ and $t\in[0,1]$, then the following inequality holds,

$$\frac{2}{h\left(\frac{1}{2}\right)}f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)g(x)\,\mathrm{d}x$$
$$\leqslant \left[mg\left(\frac{b}{m}\right)f(a) + mg\left(\frac{b}{m}\right)f(b) + f(a)g(a)\right]\int_0^1 h(t)\,\mathrm{d}t +$$
$$\left[m^2g\left(\frac{a}{m^2}\right)f(a) + m^2g\left(\frac{a}{m^2}\right)f(b) + mg\left(\frac{b}{m}\right)f(a)\right]\int_0^1 h(1-t)\,\mathrm{d}t + [f(b)-f(a)]\times$$
$$\int_0^1\left[g(a)th(t) + mg\left(\frac{b}{m}\right)th(1-t)\right]\mathrm{d}t. \tag{14}$$

Proof We can write

$$\frac{a+b}{2} = \frac{ta+(1-t)b}{2} + \frac{(1-t)a+tb}{2}.$$

Since f is convex and nonnegative on $[0,\infty)$, and g is (h,m)-convex and nonnegative on $[0,\infty)$, so we have

$$f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)$$
$$= f\left(\frac{ta+(1-t)b}{2} + \frac{(1-t)a+tb}{2}\right)\times g\left(\frac{ta+(1-t)b}{2} + \frac{(1-t)a+tb}{2}\right)$$
$$\leqslant \frac{h\left(\frac{1}{2}\right)}{2}\left[f(ta+(1-t)b) + f((1-t)a+tb)\right]\times\left[g(ta+(1-t)b) + mg\left(\frac{(1-t)a}{m}+\frac{tb}{m}\right)\right]$$
$$\leqslant \frac{h\left(\frac{1}{2}\right)}{2}\Big\{f(ta+(1-t)b)g(ta+(1-t)b) + m[tf(a)+(1-t)f(b)]g\left(\frac{(1-t)a}{m}+\frac{tb}{m}\right) +$$
$$[(1-t)f(a)+tf(b)]\left[h(t)g(a)+mh(1-t)g\left(\frac{b}{m}\right)\right] + m[(1-t)f(a)+tf(b)]g\left(\frac{(1-t)a}{m}+\frac{tb}{m}\right)\Big\}$$
$$\leqslant \frac{h\left(\frac{1}{2}\right)}{2}\Big\{f(ta+(1-t)b)g(ta+(1-t)b) + m[f(a)+f(b)]\times\left[h(t)g\left(\frac{b}{m}\right)+mh(1-t)g\left(\frac{a}{m^2}\right)\right] +$$
$$f(a)\left[g(a)h(t)+mg\left(\frac{b}{m}\right)h(1-t)\right] + [f(b)-f(a)]\left[g(a)th(t)+mg\left(\frac{b}{m}\right)th(1-t)\right]\Big\}$$
$$= \frac{h\left(\frac{1}{2}\right)}{2}\Big\{f(ta+(1-t)b)g(ta+(1-t)b) + \left[mg\left(\frac{b}{m}\right)f(a)+mf\left(\frac{b}{m}\right)f(b)+f(a)g(a)\right]h(t) +$$
$$\left[m^2g\left(\frac{a}{m^2}\right)f(a)+m^2g\left(\frac{a}{m^2}\right)f(b) + mg\left(\frac{b}{m}\right)f(a)\right]h(1-t) + [f(b)-f(a)]\left[g(a)th(t)+mg\left(\frac{b}{m}\right)th(1-t)\right]\Big\}.$$

Integrating both side of the above inequality on $[0,1]$, with respect to t, and by the fact that

$$\int_0^1 f(ta+(1-t)b)g(ta+(1-t)b)\,\mathrm{d}t = \frac{1}{b-a}\int_a^b f(x)g(x)\,\mathrm{d}x,$$

we obtain

$$f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)$$
$$\leqslant \frac{h\left(\frac{1}{2}\right)}{2}\Big\{\frac{1}{b-a}\int_a^b f(x)g(x)\,\mathrm{d}x + \left[mg\left(\frac{b}{m}\right)f(a)+mg\left(\frac{b}{m}\right)f(b)+f(a)g(a)\right]\int_0^1 h(t)\,\mathrm{d}t +$$
$$\left[m^2g\left(\frac{a}{m^2}\right)f(a)+m^2g\left(\frac{a}{m^2}\right)f(b) + mg\left(\frac{b}{m}\right)f(a)\right]\int_0^1 h(1-t)\,\mathrm{d}t +$$
$$[f(b)-f(a)]\int_0^1\left[g(a)th(t)+mg\left(\frac{b}{m}\right)th(1-t)\,\mathrm{d}t\right],$$

which completes the proof.

Corollary 2.2 Let $f,g:[0,\infty)\to[0,\infty)$, be convex functions such that $fg\in L^1([a,b])$ with $a,b\in[0,\infty)$, $a<b$, then the following inequality holds,

$$4f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)$$
$$\leqslant \frac{1}{b-a}\int_a^b f(x)g(x)\,\mathrm{d}x + \frac{2}{3}M(a,b) + \frac{5}{6}N(a,b) \tag{15}$$

where $M(a,b) = f(a)g(a) + f(b)g(b)$, $N(a,b) = f(a)g(b) + f(b)g(a)$.

Proof From Theorem 2.3 let $h(t) = t$, $m = 1$, so Corollary 2.2 immediately holds.

Theorem 2.4 Let $h_1, h_2: [0,1] \to (0,\infty)$, $f, g: [0,\infty) \to \mathbb{R}$, be functions such that $h_1 h_2 \in L^1([0,1])$, $f\,g \in L^1([\min\{\frac{a}{m_1}, \frac{a}{m_2}\}, \max\{\frac{b}{m_1}, \frac{b}{m_2}\}])$ with $a, b \in [0,\infty)$, $a < b$ and $m_1, m_2 \in (0,1]$. If f is (h_1, m_1)-convex and nonnegative on $[0,\infty)$, and if g is (h_2, m_2)-convex and nonnegative on $[0,\infty)$ with $t \in (0,1]$, then the following inequality holds,

$$\frac{1}{h_1(\frac{1}{2})h_2(\frac{1}{2})} f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)$$
$$\leqslant \frac{1}{b-a}\int_a^b f(x)g(x)\,\mathrm{d}x +$$
$$m_1 m_2 \int_0^1 f\left(\frac{(1-t)a}{m_1} + \frac{tb}{m_1}\right)g\left(\frac{(1-t)a}{m_2} + \frac{tb}{m_2}\right)\mathrm{d}t +$$
$$m_1 N_1(a,b)\int_0^1 h_1(t)h_2(t)\,\mathrm{d}t +$$
$$m_1 m_2 N_2(a,b)\int_0^1 h_1(1-t)h_2(1-t)\,\mathrm{d}t +$$
$$m_1 M_1(a,b)\int_0^1 h_1(1-t)h_2(t)\,\mathrm{d}t +$$
$$m_2 M_2(a,b)\int_0^1 h_1(t)h_2(1-t)\,\mathrm{d}t, \tag{16}$$

when

$$N_1(a,b) = m_1 f\left(\frac{b}{m_1}\right)g(a) + m_2 f(a)g\left(\frac{b}{m_2}\right),$$
$$N_2(a,b) = m_2 f\left(\frac{b}{m_1}\right)g\left(\frac{a}{m_2^2}\right) + m_1 f\left(\frac{a}{m_1^2}\right)g\left(\frac{b}{m_2}\right),$$
$$M_1(a,b) = m_1 f\left(\frac{a}{m_1^2}\right)g(a) + m_2 f\left(\frac{b}{m_1}\right)g\left(\frac{b}{m_2}\right),$$
$$M_2(a,b) = m_2 f(a)g\left(\frac{a}{m_2^2}\right) + m_1 f\left(\frac{b}{m_1}\right)g\left(\frac{b}{m_2}\right).$$

Proof We can write

$$\frac{a+b}{2} = \frac{ta+(1-t)b}{2} + \frac{(1-t)a+tb}{2}.$$

Since f is (h_1, m_1)-convex and nonnegative on $[0,\infty)$, and g is (h_2, m_2)-convex and nonnegative on $[0,\infty)$, so we have

$$f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)$$
$$= f\left(\frac{ta+(1-t)b}{2} + \frac{(1-t)a+tb}{2}\right) \times$$
$$g\left(\frac{ta+(1-t)b}{2} + \frac{(1-t)a+tb}{2}\right)$$
$$\leqslant h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\left[f(ta+(1-t)b) + m_1 f\left(\frac{(1-t)a}{m_1} + \frac{tb}{m_1}\right)\right] \times$$
$$\left[g(ta+(1-t)b) + m_2 g\left(\frac{(1-t)a}{m_2} + \frac{tb}{m_2}\right)\right]$$
$$\leqslant h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\left\{f(ta+(1-t)b)g(ta+(1-t)b) + m_1 m_2 f\left(\frac{(1-t)a}{m_1} + \frac{tb}{m_1}\right)g\left(\frac{(1-t)a}{m_2} + \frac{tb}{m_2}\right) + \right.$$
$$m_2\left[\left(h_1(t)f(a) + m_1 h_1(1-t)f\left(\frac{b}{m_1}\right)\right)\right] \times$$
$$\left[g\left(\frac{(1-t)a}{m_2} + \frac{tb}{m_2}\right)\right] + m_1 f\left(\frac{(1-t)a}{m_1} + \frac{tb}{m_1}\right) \times$$
$$\left.\left[h_2(t)g(a) + m_2 h_2(1-t)g\left(\frac{b}{m_2}\right)\right]\right\}$$
$$\leqslant h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\left\{f(ta+(1-t)b)g(ta+(1-t)b) + m_1 m_2 f\left(\frac{(1-t)a}{m_1} + \frac{tb}{m_1}\right)g\left(\frac{(1-t)a}{m_2} + \frac{tb}{m_2}\right) + \right.$$
$$m_2\left[\left(h_1(t)f(a) + m_1 h_1(1-t)f\left(\frac{b}{m_1}\right)\right)\right] \times$$
$$\left[h_2(t)g\left(\frac{b}{m_2}\right) + m_2 h_2(1-t)g\left(\frac{a}{m_2^2}\right)\right] +$$
$$m_1\left[h_1(t)f\left(\frac{b}{m_1}\right) + m_1 h_1(1-t)f\left(\frac{a}{m_1^2}\right)\right] \times$$
$$\left.\left[h_2(t)g(a) + m_2 h_2(1-t)g\left(\frac{b}{m_2}\right)\right]\right\}$$
$$= h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\left\{f(ta+(1-t)b)g(ta+(1-t)b) + m_1 m_2 f\left(\frac{(1-t)a}{m_1} + \frac{tb}{m_1}\right)g\left(\frac{(1-t)a}{m_2} + \frac{tb}{m_2}\right) + \right.$$
$$\left[m_1 f\left(\frac{b}{m_1}\right)g(a) + m_2 f(a)g\left(\frac{b}{m_2}\right)\right]h_1(t)h_2(t) +$$
$$\left[m_2^2 f(a)g\left(\frac{a}{m_2^2}\right) + m_1 m_2 f\left(\frac{b}{m_1}\right)g\left(\frac{b}{m_2}\right)\right]h_1(t)h_2(1-t) +$$
$$\left[m_1 m_2 f\left(\frac{b}{m_1}\right)g\left(\frac{b}{m_2}\right) + m_1^2 f\left(\frac{a}{m_1^2}\right)g(a)\right]h_1(1-t)h_2(t) +$$
$$\left[m_1 m_2^2 f\left(\frac{b}{m_1}\right)g\left(\frac{a}{m_2^2}\right)\right] +$$

$m_1^2 m_2 f(\frac{a}{m_1^2}) g(\frac{b}{m_2})] h_1(1-t) h_2(1-t)\}.$

Integrating both side of the above inequality on [0, 1], with respect to t, and by the fact that

$$\int_0^1 f(ta+(1-t)b) g(ta+(1-t)b)\,\mathrm{d}t = \frac{1}{b-a}\int_a^b f(x)g(x)\,\mathrm{d}x,$$

we obtain

$$\begin{aligned}
&f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) \\
&\leqslant h_1\left(\frac{1}{2}\right) h_2\left(\frac{1}{2}\right)\left\{\frac{1}{b-a}\int_a^b f(x)g(x)\,\mathrm{d}x + \right. \\
&m_1 m_2 \int_0^1 f\left(\frac{(1-t)a}{m_1}+\frac{tb}{m_1}\right) g\left(\frac{(1-t)a}{m_2}+\frac{tb}{m_2}\right)\mathrm{d}t + \\
&\left[m_1 f\left(\frac{b}{m_1}\right) g(a) + m_2 f(a) g\left(\frac{b}{m_2}\right)\right]\int_0^1 h_1(t)h_2(t)\,\mathrm{d}t + \\
&m_2\left[m_2 f(a) g\left(\frac{a}{m_2^2}\right) + m_1 f\left(\frac{b}{m_1}\right) g\left(\frac{b}{m_2}\right)\right]\times \\
&\int_0^1 h_1(t)h_2(1-t)\,\mathrm{d}t + m_1\left[m_2 f\left(\frac{b}{m_1}\right) g\left(\frac{b}{m_2}\right) + \right. \\
&\left. m_1 f\left(\frac{a}{m_1^2}\right) g(a)\right]\int_0^1 h_1(1-t)h_2(t)\,\mathrm{d}t,
\end{aligned}$$

which completes the proof.

Remark 2.5 If we choose $m_1 = m_2 = 1, h_1(t) = t, h_2(t) = t^s$ in (16) for some $s \in (0,1)$, then we obtain the inequaltiy (10).

Remark 2.6 If we choose $m_1 = m_2 = 1$ and $h_1(t) = h_2(t) = t$ in (16), we can obtain the inequality (7).

Corollary 2.3 Let $f:[0,\infty)\to\mathbb{R}$, be an (h,m)-convex and nonnegative on $[0,\infty)$ with $m\in(0,1]$ and $t\in[0,1]$. If $h\in L^1([0,1])$, $f\in L^1([a,b])$ with $a,b\in[0,\infty)$, $a<b$, then the following inequality holds,

$$\begin{aligned}
&\frac{2}{h\left(\frac{1}{2}\right)} f\left(\frac{a+b}{2}\right) \leqslant \\
&\frac{1}{b-a}\left[\int_a^b f(x)\mathrm{d}x + m\int_a^b f\left(\frac{x}{m}\right)\mathrm{d}x\right] + \left[f(a) + m f\left(\frac{b}{m}\right)\right]\times \\
&\int_0^1 h(t)\,\mathrm{d}t + m\left[f\left(\frac{b}{m}\right) + m f\left(\frac{a}{m^2}\right)\right]\int_0^1 h(1-t)\,\mathrm{d}t.
\end{aligned} \tag{17}$$

Proof We choose $g(x) = 1$ for all $x\in[a, b]$, and $h_2(t) = t, m_2 = 1$ in (16), then we can obtain the following inequality

$$\begin{aligned}
&\frac{2}{h_1\left(\frac{1}{2}\right)} f\left(\frac{a+b}{2}\right) \\
&\leqslant \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x + m_1\int_0^1 f\left(\frac{(1-t)a}{m_1}+\frac{tb}{m_1}\right)\mathrm{d}t + \\
&\left[f(a) + m_1 f\left(\frac{b}{m_1}\right)\right]\int_0^1 h_1(1-t)\,\mathrm{d}t + \\
&m_1\left[f\left(\frac{b}{m_1}\right) + m_1 f\left(\frac{a}{m_1^2}\right)\right]\int_0^1 h_1(1-t)\,\mathrm{d}t.
\end{aligned}$$

By the fact that

$$\int_0^1 f\left(\frac{(1-t)a}{m_1}+\frac{tb}{m_1}\right)\mathrm{d}t = \frac{1}{b-a}\int_a^b f\left(\frac{x}{m_1}\right)\mathrm{d}x,$$

we can obtain the result.

Remark 2.7 If in Corollary 2.3 we choose $h(t) = t^s$, $m = 1$, we can obtain the following inequality for s-convex functions,

$$2^s f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x + \frac{f(a)+f(b)}{s+1},$$

which is the Remark 4 in Ref. [11].

Theorem 2.5 Let $h_1, h_2:[0,1]\to(0,\infty)$, $f,g:[0,\infty)\to\mathbb{R}$, be functions such that $h_1h_2\in L^1([0,1])$, $fg\in L^1([a,b])$ with $a,b\in[0,\infty)$, $a<b$. If f is (h_1,m_1)-convex and nonnegative on $[0,\infty)$, and if g is (h_2,m_2)-convex and nonnegative on $[0,\infty)$ with $m_1,m_2\in(0,1]$ and $t\in[0,1]$, then the following inequality holds,

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b f(x)g(x)\,\mathrm{d}x \\
&\leqslant f(a) P(h_1(t), h_2(t), h_2(1-t)) + \\
&m_1 f\left(\frac{b}{m_1}\right) P(h_1(1-t), h_2(t), h_2(1-t)),
\end{aligned} \tag{18}$$

where

$$\begin{aligned}
&P(h_1(t), h_2(t), h_2(1-t)) \\
&= g(a)\int_0^1 h_1(t)h_2(t)\,\mathrm{d}t + m_2 g\left(\frac{b}{m_2}\right)\int_0^1 h_1(t)h_2(1-t)\,\mathrm{d}t, \\
&P(h_1(1-t), h_2(t), h_2(1-t)) \\
&= g(a)\int_0^1 h_1(1-t)h_2(t)\,\mathrm{d}t + \\
&m_2 g\left(\frac{b}{m_2}\right)\int_0^1 h_1(1-t)h_2(1-t)\,\mathrm{d}t.
\end{aligned}$$

Proof From the proof of Theorem 2.2, we are easy to obtain the inequality (18).

Remark 2.8 If we choose $m_1 = m_2 = 1$, $h_1(t) = t$ and $h_2(t) = t^s$ in (18), we can obtain the inequality (8).

Remark 2.9 If we choose $m_1 = m_2 = 1$, $h_1(t) = t^{s_1}$ and $h_2(t) = t^{s_2}$ in (18), we can obtain the inequality (9).

Remark 2.10 If we choose $m_1 = m_2 = 1$, $h_1(t) = t$ and $h_2(t) = t$ in (18), we can obtain the inequality (6).

Remark 2.11 From Remark 1.1, if we choose proper values of m_i and $h_i(t)$ $(i = 1,2)$ in inequalities (12), (16) and (18), we can botain the corresponding inequalities under the condition of different convexity. For example,

1) If we choose $h_1(t) = h_2(t) = t$ in (16) and (18), we can obtain the following Hadamard-type inequalities for products of m-convex function,

$$4f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)g(x)\,\mathrm{d}x + m_1m_2\int_0^1 f\left(\frac{(1-t)a}{m_1} + \frac{tb}{m_1}\right)g\left(\frac{(1-t)a}{m_2} + \frac{tb}{m_2}\right)\mathrm{d}t + \frac{1}{3}[N_1(a,b) + m_1m_2N_2(a,b)] + \frac{1}{6}[m_1M_1(a,b) + m_2M_2(a,b)], \quad (19)$$

and

$$\frac{1}{b-a}\int_a^b f(x)g(x)\,\mathrm{d}x \leqslant \frac{1}{3}\left[f(a)g(a) + m_1m_2 f\left(\frac{b}{m_1}\right)g\left(\frac{b}{m_2}\right)\right] + \frac{1}{6}N_1(a,b), \quad (20)$$

where $N_1(a,b)$, $N_2(a,b)$, $M_1(a,b)$, $M_2(a,b)$ are as in Theorem 2.4.

2) If we choose $m_1 = m$, $h_1(t) = t$; $m_2 = 1$, $h_2(t) = t^s$ in (16) and (18), we can obtain the Hadamard-type inequalities for products of m-convex function and s-convex function as follows,

$$2^{s+1}f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)g(x)\,\mathrm{d}x + m\int_0^1 f\left(\frac{(1-t)a}{m} + \frac{tb}{m}\right)g((1-t)a + tb)\,\mathrm{d}t + \frac{1}{s+2}[N_1'(a,b) + mN_2'(a,b)] + mM_1'(a,b)B(s+1,2) + M_2'(a,b)B(2,s+1), \quad (21)$$

and

$$\frac{1}{b-a}\int_a^b f(x)g(x)\,\mathrm{d}x \leqslant \frac{1}{s+2}N_1'(a,b) + f(a)g(b)B(2,s+1) + mf\left(\frac{b}{m}\right)g(a)B(s+1,2), \quad (22)$$

where $B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,\mathrm{d}t$, and

$$N_1'(a,b) = mf\left(\frac{b}{m}\right)g(a) + f(a)g(b),$$

$$N_2'(a,b) = f\left(\frac{b}{m}\right)g(a) + mf\left(\frac{a}{m^2}\right)g(b),$$

$$M_1'(a,b) = mf\left(\frac{b}{m^2}\right)g(a) + f\left(\frac{b}{m}\right)g(b),$$

$$M_2'(a,b) = f(a)g(a) + mf\left(\frac{b}{m}\right)g(b).$$

References

[1] Breckner W W. Stetigkeitsaussagen füreine Klasse verallgemeinerter konvexer funktionen in topologischen linearen Raumen[J]. Pupl Inst Math, 1978, 23: 13-20.

[2] Varošanec S. On h-convexity[J]. J Math Anal Appl, 2007, 326: 303-311.

[3] Sarikaya M Z, Saglam A, Yildirim H. On some Hadamard-type inequalities for h-convex functions [J]. Journal of Mathematical Inequalities, 2008, 2(3): 335-341.

[4] Sarikaya M Z, Set E, Özdemir M E. On some new inequalities of Hadamard type involving h-convex functions [J]. Acta Math Univ Comenianae, 2010, LXXIX(2): 265-272.

[5] Dragomir S S, Toader G. Some inequalities for m – convex functions[J]. Studia Univ Babes – Bolyai Math, 1993, 38(1): 21-28.

[6] Avci M, Kavurmaci H, Özdemir M E. New inequalities of Hermite-Hadamard type via s-convex functions in the second sense with applications [J]. Applied Mathematics and Computation, 2011, 217(12): 5 171 – 5 176.

[7] Özdemir M E, Yildiz C, Akdemir A O, et al. On some inequalities for s-convex functions and applications [J]. Journal of Inequalities and Applications, 2013, 2013(1): 1-11.

[8] Özdemir M E, Akdemir A O, Set E. On (h, m)-convexity and Hadamard-type inequalities[J]. Mathematics, 2011, 3: 1-9.

[9] Dragomir S S, Fitzpatrick S. The Hadamard's inequality for s-convex functions in the second sense[J]. Demonstratio Math, 1999, 32(4): 687-696.

[10] Pachpatte B G. On some inequalities for convex functions [J]. RGMIA Res Rep Coll, 2003, 6(E): 1-9.

[11] Kirmaci U S, Bakula M K, Özdemir M E, et al. Hadamard-type inequalities for s-convex functions [J]. Appl Math Comput, 2007, 193(1): 26-35.