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Iterated Hardy-Littlewood maximal function*

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Abstract In the paper, we investigate the iterated non-centered Hardy-Littlewood maximal function and the iterated centered Hardy-Littlewood maximal function. We prove that the limit of the iterated maximal function is just a fixed point of maximal operator. As an application of the fixed point theory, we finally obtain that the fixed point is $\|f\|_\infty$ for non-centered Hardy-Littlewood maximal operator. The same is true for the centered Hardy-Littlewood maximal operator only for $n=1,2$.

Keywords Hardy-Littlewood maximal function; fixed point; iterated Hardy-Littlewood maximal function

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迭代的哈代-李特伍德极大函数

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摘　要 研究迭代的非中心型哈代-李特伍德极大函数和迭代的中心型哈代-李特伍德极大函数。证明迭代极大函数的极限是极大算子的一个不动点。作为不动点理论的一个应用,最终得到,对于非中心型哈代-李特伍德极大算子,这个不动点处处为 $\|f\|_\infty$。对于中心型哈代-李特伍德极大算子,仅在 $n=1,2$ 时有相同的结果。

关键词 哈代-李特伍德极大函数;不动点;迭代的哈代-李特伍德极大函数

Define the centered Hardy-Littlewood maximal function by

$$M_c f(x) = \sup_{r>0} \frac{1}{|B(x,r)|}\int_{B(x,r)} |f(y)|\,dy, \tag{1}$$

and the non-centered Hardy-Littlewood maximal function by

$$Mf(x) = \sup_{B\ni x} \frac{1}{|B|}\int_B |f(y)|\,dy, \tag{2}$$

where B is a ball and $B(x,r)$ is a ball with the center at the point x and the radius r. The basic real-variable construct was introduced for $n=1$ by Hardy and Littlewood[1] and for $n\geqslant 2$ by Wiener[2]. It is well-known that the Hardy-Littlewood maximal

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function plays an important role in many parts of analysis. It is a classical mean operator frequently used to majorize other important operators in harmonic analysis.

It is clear that

$$M_c f(x) \leqslant Mf(x) \leqslant 2^n M_c f(x) \qquad (3)$$

holds for all $x \in \mathbb{R}^n$. Both M and M^c are sublinear operators and the two functions Mf and $M_c f$ never vanish unless $f = 0$ almost everywhere[3]. The study of the boundedness for M or M_c is fairly complete[4]. The primary purpose of this paper is to study the properties of the iterated Hardy-Littlewood maximal function.

Let M be the non-centered Hardy-Littlewood maximal function defined by (2). Define the iterated non-centered Hardy-Littlewood maximal function denoted by M^{k+1} as follows:

$$M^{k+1}f(x) := M(M^k f)(x), \qquad (4)$$

for $k = 1, 2, \cdots$, and $x \in \mathbb{R}^n$. Set $M^1 f(x) := Mf(x)$.

In the same way, we can set

$$M_c^{k+1}f(x) := M_c(M_c^k f)(x). \qquad (5)$$

We all know that both operators M_c and M have the L^p-boundedness and the two maximal functions $M_c f$ and Mf have a little difference in the pointwise sense from inequalities (3). We want to investigate the limit of the iterated Hardy-Littlewood maximal function.

Wei et al[5] studied the limit of $M^k f$ and obtained Theorem A as follows.

Theorem A For any $f \in L^\infty(\mathbb{R}^n)$, the equation

$$\lim_{k\to\infty} M^k f(x) = \| f \|_\infty \qquad (6)$$

holds for any $x \in \mathbb{R}^n$.

For M_c, we want to know whether it has the same properties as M. Unexpectedly, the limit of $M_c^k f$ is essentially different from the limit of $M^k f$.

Now we formulate our main results as follows.

Theorem B If $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, then

$$\lim_{k\to\infty} M_c^k f(x) = \| f \|_\infty \qquad (7)$$

holds for every $x \in \mathbb{R}^n$ if and only if $n = 1, 2$.

Theorem C Let $f \in L^1_{\text{loc}}(\mathbb{R}^n)$. We have

$$\lim_{k\to\infty} M^k f(x) = \| f \|_\infty,$$

for every $x \in \mathbb{R}^n$ and any $n \in N$.

We remark that the range of function f in Theorem C is wider than that in Theorem A. Furthermore in this paper we will use some novel ideas to prove Theorem C.

1 Fixed point of Hardy-Littlewood maximal operator

To prove our main theorems, we first provide some definitions and lemmas which will be used in the follows. Some lemmas can be found in classic literatures and here we omit their proofs.

Definition 1.1 A function F is called a fixed point of a operator T, if

$$TF(x) = F(x) \qquad (8)$$

holds for all $x \in \mathbb{R}^n$.

Obviously if F is a fixed point of the operator T, then we have

$$\lim_{k\to\infty} T^k F(x) = F(x).$$

By the Lesbegue differentiation theorem, for almost all $x \in \mathbb{R}^n$, we have

$$M_c f(x) \geqslant | f(x) |$$

and

$$Mf(x) \geqslant | f(x) |.$$

For the iterated Hardy-Littlewood maximal operator, we have the following lemma.

Lemma 1.1 For $x \in \mathbb{R}^n$, and $k \geqslant 1$, the two inequalities

$$M^{k+1}f(x) \geqslant M^k f(x) \qquad (9)$$

and

$$M_c^{k+1}f(x) \geqslant M_c^k f(x) \qquad (10)$$

hold for all $f \in L^1_{\text{loc}}(\mathbb{R}^n)$.

Proof Set $E = \{x : x$ is not the Lesbegue point of $|f|\}$. It follows from the Lesbegue differentiation theorem that $m(E) = 0$. Actually we merely need to prove

$$M^2 f(x) \geqslant Mf(x)$$

for all $x \in \mathbb{R}^n$.

We conclude that

$$Mf(x) = \sup_{B \ni x} \frac{1}{|B|}\int_B | f(y) | \, dy$$

$$= \sup_{B \ni x} \frac{1}{|B|}\int_{B\setminus E} | f(y) | \, dy$$

$$= \sup_{B \ni x} \frac{1}{|B|}\int_{B\setminus E} \left\{ \lim_{r\to 0} \frac{1}{|B(y,r)|}\int_{B(y,r)} | f(u) | \, du \right\} dy$$

$$\leqslant \sup_{B \ni x} \frac{1}{|B|}\int_{B\setminus E} Mf(y)\, dy = \sup_{B \ni x} \frac{1}{|B|}\int_B Mf(y)\, dy$$

$$= M^2 f(x). \tag{11}$$

Using the completely same method, we can obtain that $M_c^2 f(x) \geqslant M_c f(x)$. □

By Lemma 1.1, since $M^k f$ monotonously increases, the limit of $M^k f(x)$ exists for all $x \in \mathbb{R}^n$.

Lemma 1.2 If $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ is a fixed point of M_c, then there exists another function $f_t \in C^\infty(\mathbb{R}^n) \cap L^1_{\mathrm{loc}}(\mathbb{R}^n)$ such that f_t is a fixed point of M_c as well.

Proof Set $\phi \in C_c^\infty(\mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^n} \phi(x)\,\mathrm{d}x = 1.$$

For $t > 0$, set

$$f_t = f * \phi_t,$$

where $\phi_t(x) = t^{-n}\phi(x/t)$, for all $x \in \mathbb{R}^n$. Obviously we have

$$f_t \in C^\infty(\mathbb{R}^n) \cap L^1_{\mathrm{loc}}(\mathbb{R}^n).$$

Put

$$\chi_r = \frac{1}{|B_r|}\chi_{B_r}.$$

The centered Hardy-Littlewood maximal function is written by

$$M_c(f)(x) = \sup_{r>0} \chi_r * |f|(x). \tag{12}$$

Note f is a fixed point of M_c. This implies $f \geqslant 0$. We have that

$$\chi_r * (\phi_t * f)(x) = \int_{\mathbb{R}^n} \phi_t(y)(\chi_r * \tau_y f)(x)\,\mathrm{d}y$$
$$\leqslant \int_{\mathbb{R}^n} \phi_t(y) M_c(\tau_y f)(x)\,\mathrm{d}y = \phi_t * M_c(f)(x)$$
$$= f_t(x). \tag{13}$$

On the other hand, it follows from (12) and (13) that $M_c(f_t)(x) = M_c(\phi_t * f)(x) = \sup_{r>0} \chi_r * |\phi_t * f|(x) \leqslant f_t(x)$. Since f_t is smooth function, it follows from the definition of $M_c f$ that

$$M_c(f_t)(x) \geqslant f_t(x).$$

Thus we obtain that f_t is a fixed point of M_c.

Using the similar method, we can easily prove that f_t is a fixed point of M if f is a fixed point of M. □

Lemma 1.3 There is a non-constant fixed point of M_c in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ if and only if there exists a non-negative upper-harmonic function.

There is a non-constant fixed point of M in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ if and only if there exists a non-negative function $f \in \mathbb{C}^\infty(\mathbb{R}^n) \cap L^1_{\mathrm{loc}}(\mathbb{R}^n)$ such that $f(x) \geqslant Mf(x)$ for all $x \in \mathbb{R}^n$.

Lemma 1.3 is due to that for a smooth function, every point in $\mathbb{R}^n$ is its Lesbegue point. We only need that $f(x) \geqslant Mf(x)$ to guarantee that the function is a fixed point.

Lemma 1.4 If f is a non-constant and smooth function, and f is a fixed point of M, then, in any closed ball, the minimum value of f is gotten only in the sphere.

Lemma 1.4 has the same proof as the proof of extremism principle of harmonic function. For the details please see Ref. [6].

Lemma 1.5 Suppose that f is a fixed point of M. If $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, then we have $f = C \leqslant \infty$; if $f \notin L^1_{\mathrm{loc}}(\mathbb{R}^n)$, then we have $f(x) = \infty$.

Proof. Since f is a fixed point of M, it follows from Lemma 1.2 that there exists $f_t \in \mathbb{C}^\infty(\mathbb{R}^n) \cap L^1_{\mathrm{loc}}(\mathbb{R}^n)$ such that f_t is a fixed point of M as well.

Suppose that B is a ball in $\mathbb{R}^n$ and $f_t \in \mathbb{C}^\infty(\mathbb{R}^n) \cap L^1_{\mathrm{loc}}(\mathbb{R}^n)$ such that $Mf_t(x) = f_t(x)$ for all $x \in \mathbb{R}^n$. We use the proof by contradiction.

If f_t is not a constant, then, by Lemma 1.4, there is at least one point $x \in \partial B$ such that $f_t(y) > f_t(x)$ holds for all $y \in B^\circ$.

Note that f_t is a fixed point of M. Thus we have that

$$f_t(x) < \frac{1}{m(B^\circ)}\int_{B^\circ} |f_t(y)|\,\mathrm{d}y$$
$$= \frac{1}{m(B)}\int_B |f_t(y)|\,\mathrm{d}y \leqslant Mf_t(x) = f_t(x). \tag{14}$$

This is impossible. Consequently it implies that $f_t(x) = C$ for all $x \in \mathbb{R}^n$.

Next we will prove that $f(x) = C$ for all $x \in \mathbb{R}^n$.

Choose a radial nonnegative function $\phi \in C_c^\infty(\mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^n} \phi(x)\,\mathrm{d}x = 1,\ \operatorname{supp}\phi \subset \{x \in \mathbb{R}^n : |x| \leqslant 1\}$$

and $\phi(x) \geqslant \phi(x')$ for $0 \leqslant |x| \leqslant |x'|$.

For each $t > 0$, we have $f_t(x) = C$.

Set $B_R = \{x \in \mathbb{R}^n : |x| < R\}$, for $R > 0$. We conclude from Lemma 1.2 that

$$C = f_t(x) = \int_{\mathbb{R}^n} (f\chi_{B_R} + f\chi_{\mathbb{R}^n \setminus B_R})(x-y)\phi_t(y)\,\mathrm{d}y$$

$$= f\chi_{B_R} * \phi_t(x) + f\chi_{\mathbb{R}^n\setminus B_R} * \phi_t(x). \quad (15)$$

It follows from (15) that

$$\lim_{t\to0} f\chi_{B_R} * \phi_t(x) + \lim_{t\to0} f\chi_{\mathbb{R}^n\setminus B_R} * \phi_t(x) = C. \quad (16)$$

Note that $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Thus we have $f\chi_{B_R} \in L^1(\mathbb{R}^n)$. This implies that

$$\lim_{t\to0} f\chi_{B_R} * \phi_t(x) = f\chi_{B_R}(x) \quad (17)$$

for almost every $x \in \mathbb{R}^n$. By the property of convolution, we get that

$$\operatorname{supp} f\chi_{\mathbb{R}^n\setminus B_R} * \phi_t \subset \{x \in \mathbb{R}^n : |x| \geqslant R - t\}. \quad (18)$$

Combing (16), (17) with (18) yields that

$$f\chi_{B_R}(x) + \lim_{t\to0} f\chi_{\mathbb{R}^n\setminus B_R} * \phi_t(y) = C$$

holds for almost every $x \in \mathbb{R}^n$. This is equivalent to that

$$f\chi_{B_R}(x) = C$$

holds for almost every $x \in \mathbb{R}^n$. Let $R \to \infty$, then we have

$$f(x) = C$$

for almost every $x \in \mathbb{R}^n$. This implies that $Mf(x) = C$ for every $x \in \mathbb{R}^n$. Note that f is a fixed point of M, that is,

$$Mf(x) = f(x).$$

Thus we must obtain that

$$f(x) = C$$

for every $x \in \mathbb{R}^n$.

If $f \notin L^1_{\mathrm{loc}}(\mathbb{R}^n)$, then there is a ball B such that

$$\int_B |f(x)|\,dx = \infty.$$

We have $Mf(x) = \infty$. So we have $f(x) = \infty$ for every $x \in \mathbb{R}^n$. □

We remark that M_c has essential difference with M with respect to the fixed point. We all know that when $n \geqslant 3$, the function $f(x) = |x|^{2-n}$ is a harmonic function in $\mathbb{R}^n\setminus\{0\}$. In fact, we can easily check that $f(x) = |x|^{2-n}$ is a fixed point of M_c.

Korry[7] obtained the following lemma 1.6.

Lemma 1.6 For the M_c, if $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ with $n = 1, 2$ and $M_c f = f$, then we have $f = C \geqslant 0$.

2 Main theorems

In the section, for any local integral function, the limit of the iterated Hardy-Littlewood maximal function is a fixed point of Hardy-Littlewood maximal operator.

Theorem 2.1 Write

$$\lim_{k\to\infty} M^k f(x) = F(x).$$

We have

$$MF(x) = F(x).$$

That is, F a fixed point of M.

In the same way, write

$$\lim_{k\to\infty} M_c^k f(x) = F_c(x),$$

then $M_c F_c(x) = F_c(x)$. That is, F_c a fixed point of M_c.

Proof We only prove the first part of Theorem 2.1. It follows that

$$MF(x) = M \lim_{k\to\infty} M^k f(x) = \sup_{B\ni x} \frac{1}{m(B)} \int_B \lim_{k\to\infty} M^k f(y)\,dy. \quad (19)$$

Associate to an arbitrary $\varepsilon > 0$ $B_\varepsilon \ni x$ such that

$$\begin{aligned} MF(x) - \varepsilon &\leqslant \frac{1}{m(B_\varepsilon)} \int_{B_\varepsilon} \lim_{k\to\infty} M^k f(y)\,dy \\ &\leqslant \lim_{k\to\infty} \frac{1}{m(B_\varepsilon)} \int_{B_\varepsilon} M^k f(y)\,dy \\ &\leqslant \lim_{k\to\infty} M^{k+1} f(x) = F(x) \\ &\leqslant MF(x). \end{aligned} \quad (20)$$

That is

$$MF(x) = F(x). \quad (21)$$

□

Lemma 2.1 Write

$$\lim_{k\to\infty} M^k f(x) = F(x).$$

If $F(x) = C$ for all $x \in \mathbb{R}^n$, then we have $C = \|f\|_\infty$.

Proof Since $F(x) = C$, it implies from the definition of Hardy-Littlewood maximal function that $C \leqslant \|f\|_\infty$. By the definition of essential supremum of function, associate to an arbitrary $\varepsilon > 0$, a set $E \subset \mathbb{R}^n$ with $m(E) > 0$, such that

$$|f(x)| > \|f\|_\infty - \varepsilon,$$

for $x \in E$. When $x \in E$ is the lebesgue point of f, we have that

$$C = F(x) \geqslant Mf(x) \geqslant |f(x)| > \|f\|_\infty - \varepsilon.$$

By the arbitrary property of ε, we immediately have

$$C \geqslant \|f\|_\infty.$$

Consequently we have $C = \|f\|_\infty$ □

Next we will prove our main theorems.

The proof of Theorem B

Proof It follows from Theorem 2.1 that

$$F_c(x) = \lim_{k\to\infty} M_c^k f(x) \tag{22}$$

is a fixed point of M_c.

By Lemm 1.6, if $F_c \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ with $n = 1, 2$, then $F_c = C$.

Since $F_c(x) = C$, it implies from the definition of center Hardy-Littlewood maximal function that $C \leqslant \|f\|_\infty$.

By the definition of essential supremum of function, associate to an arbitrary $\varepsilon > 0$ a set $E \subset \mathbb{R}^n$ with $m(E) > 0$, such that

$$|f(x)| > \|f\|_\infty - \varepsilon,$$

for $x \in E$. Since $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, almost every points in $\mathbb{R}^n$ is the lebesgue point of f. Choose $x \in E$ is the lebesgue point of f. We have that

$$C = F(x) \geqslant M_c f(x) \geqslant |f(x)| > \|f\|_\infty - \varepsilon.$$

By the arbitrary property of ε, we immediately have

$$C \geqslant \|f\|_\infty.$$

Consequently we have $F_c = \|f\|_\infty$.

If $F_c \notin L^1_{\mathrm{loc}}(\mathbb{R}^n)$, we can easily have $F = \infty = \|f\|_\infty$. □

The proof of Theorem C

Proof It follows from Theorem 2.1 that

$$F(x) = \lim_{k\to\infty} M^k f(x) \tag{23}$$

is a fixed point of M.

By Lemma 1.5, if $F \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, then we have $F = C$.

It follows from Lemma 2.1 that $F = \|f\|_\infty$.

If $F \notin L^1_{\mathrm{loc}}(\mathbb{R}^n)$, then we have $\|f\|_\infty = \infty$, and there exists a ball $B(0,R)$ such that

$$\int_{B(0,R)} |F(x)| \, dx = \infty.$$

Note that F a fixed point of M. We have

$$F(x) = MF(x) = \infty.$$

Consequently, we have $F = \|f\|_\infty$. □

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