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$\mathbb{H}P^3$ 中共形极小曲面的几何 *

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摘　要　通过扭映射 $\pi:\mathbb{C}P^7\to\mathbb{H}P^3$ 构造出 $\mathbb{H}P^3$ 中曲率为 4/7,4/19,4/27 的 6 个共形极小二维球面的例子. 由于扭映射 $\pi:\mathbb{C}P^{2n+1}\to\mathbb{H}P^n$ 给出了 $\mathbb{C}P^{2n+1}$ 的水平极小曲面与 $\mathbb{H}P^n$ 中极小曲面的一个自然等同,利用 Bolton 等得出的在 $\mathbb{C}P^n$ 中常曲率共形极小二维球面的结论,根据 CHEN Xiaodong 和 JIAO Xiaoxiang 给出的 $\mathbb{H}P^n$ 中常曲率共形极小二维球面的一般方法,构造出 $\mathbb{H}P^n$ 中的共形极小曲面的例子.

关键词　四元数射影空间;共形极小曲面;扭映射

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The geometry of conformal minimal surfaces in $\mathbb{H}P^3$

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Abstract　In this work, we construct six examples of conformal minimal two-spheres surfaces with constant curvatures of 4/7, 4/19, and 4/27 by the twistor map $\pi:\mathbb{C}P^7\to\mathbb{H}P^3$. By the twistor map, we know that the horizontal minimal surfaces in $\mathbb{C}P^{2n+1}$ are equivalent to the minimal surfaces in $\mathbb{H}P^n$, and many conclusions about conformal minimal two-spheres with constant curvatures in $\mathbb{C}P^n$ have been given. Based on Bolton's conclusion and Chen's general method to construct conformal minimal two-spheres, we get some conformal minimal two-sphere surfaces in $\mathbb{H}P^3$.

Keywords　quaternionic projective space; conformal minimal surface; twistor map

对极小曲面的研究一直是微分几何研究领域中的一个重要课题,特别是关于极小曲面的几何性质以及分类问题的研究。当外围空间是空间形式的时候,一些重要的分类结果已经先后被提出。更为一般的情况是外围空间是对称空间的时候,对称空间中的极小曲面研究也有不错的进展,但依旧有许多值得关注的问题。所以,本文主要关注外围空间是四元素射影空间的情形,这对于四元素射影空间中极小曲面的几何及分类的研究是具有重要意义的。

在近几十年中,国内外对极小曲面的研究都取得了许多重要成果。

1982 年, Bryant[1] 通过扭映射 $\pi:\mathbb{C}P^3\to\mathbb{H}P^1$ 证明 $\mathbb{C}P^3$ 中水平全纯曲面的投影为 $\mathbb{H}P^1$ 中的极小

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曲面，并且构造出$\mathbb{C}P^3$ 中紧致的非分歧的全纯水平曲面。1986 年 Aithal[2] 构造$\mathbb{H}P^2$ 中所有的调和二维球面。Bolton 等[3] 也在 1988 年给出$\mathbb{C}P^n$ 中的 Veronese 序列即常曲率极小二维球面。1991 年，Bahy-El-Dien 和 Wood[4] 构造$\mathbb{H}P^n$ 中的所有的调和二维球面。2014 年，He 和 Jiao[5] 给出$\mathbb{H}P^2$ 中线性满、全非分歧的常曲率共形极小球面的分类。同年，He 和 Jiao[6] 给出$\mathbb{H}P^n$ 中第二基本形式平行的共形极小球面的分类。

近几十年中，关于复射影空间中极小曲面的几何研究，许多学者已经给出很多重要的结论。而由扭映射 $\pi:\mathbb{C}P^3 \to \mathbb{H}P^1$ 可知四元素射影空间与复射影空间的水平分量有一个自然等同，所以我们希望通过复射影空间研究四元素射影空间。

1 预备知识

首先介绍四元数、扭映射以及相关的一些知识。

四元数$\mathbb{H}$ 是以 1，i，j，k 为基的一个四维实向量空间，即$\mathbb{H} = \{a + b\mathrm{i} + c\mathrm{j} + d\mathrm{k} \mid a,b,c,d \in \mathbb{R}\}$，其中 1，i，j，k 满足：

$$\mathrm{i}^2 = \mathrm{j}^2 = \mathrm{k}^2 = -1, \mathrm{ij} = \mathrm{k} = -\mathrm{ji},$$
$$\mathrm{jk} = \mathrm{i} = -\mathrm{kj}, \mathrm{ki} = \mathrm{j} = -\mathrm{ik}.$$

由此可见$\mathbb{H}$ 为一个不可交换环。

因而，对于 $h_1 = a_1 + b_1\mathrm{i} + c_1\mathrm{j} + d_1\mathrm{k}, h_2 = a_2 + b_2\mathrm{i} + c_2\mathrm{j} + d_2\mathrm{k} \in \mathbb{H}$ 有

$$\begin{aligned} h_1h_2 = {} & (a_1a_2 - b_1b_2 - c_1c_2 - d_1d_2) + \\ & (a_1b_2 + a_2b_1 + c_1d_2 - c_2d_1)\mathrm{i} + \\ & (a_1c_2 + a_2c_1 + d_1b_2 - d_2b_1)\mathrm{j} + \\ & (a_1d_2 + a_2d_1 + b_1c_2 - b_2c_1)\mathrm{k}. \end{aligned}$$

与复数$\mathbb{C}$ 类似，四元数上也有个自然的共轭作用 *：若 $h = a + b\mathrm{i} + c\mathrm{j} + d\mathrm{k}$，则 $h^* = a - b\mathrm{i} - c\mathrm{j} - d\mathrm{k}$.

更为经常的，我们是把$\mathbb{H}$ 看成以 1，j 为基的复数域$\mathbb{C}$ 上的二维右模，则 $\forall h \in \mathbb{H}$，可写成 $\boldsymbol{h} = \boldsymbol{u} + \mathrm{j}\boldsymbol{v}, \boldsymbol{u}, \boldsymbol{v} \in \mathbb{C}$。对于

$\boldsymbol{h}_1 = \boldsymbol{u}_1 + \mathrm{j}\boldsymbol{v}_1, \boldsymbol{h}\boldsymbol{h}_1 = (\boldsymbol{u}\boldsymbol{v} - \bar{\boldsymbol{u}}_1\boldsymbol{v}_1) + \mathrm{j}(\boldsymbol{u}_1\boldsymbol{v} + \bar{\boldsymbol{u}}\boldsymbol{v}_1)$，且知上述两种定义等价。

一般来说，用$\mathbb{H}^n$表示 n 个四元数组成的列向量，$\mathbb{H}P^n$ 表示$\mathbb{H}^{n+1}$中所有通过原点的直线组成的集合，则$[\boldsymbol{h}_1]_{\mathbb{H}}$ 和$[\boldsymbol{h}_2]_{\mathbb{H}} \in \mathbb{H}P^n$，$[\boldsymbol{h}_1]_{\mathbb{H}} = [\boldsymbol{h}_2]_{\mathbb{H}}$ 当且仅当存在 $\boldsymbol{h} \in \mathbb{H}$，使得 $\boldsymbol{h}_1 = \boldsymbol{h}_2\boldsymbol{h}$。

若令 $Sp(n) = \{\boldsymbol{H} \in GL(n;\mathbb{H}) \mid \boldsymbol{H}^{*\mathrm{T}} \cdot \boldsymbol{H} = \boldsymbol{I}_n\}$，其中 $\boldsymbol{I}_n$ 表示 n 阶单位矩阵，可在$\mathbb{H}P^n$ 定义一个作用使得$\mathbb{H}P^n$ 有齐性表示

$$\mathbb{H}P^n = Sp(n+1)/(Sp(1) \times Sp(n)).$$

根据文献[7]可知有如下的一个交换图：

$$\begin{tikzcd} & Sp(n+1) \arrow[dl, "\tau_2"'] \arrow[dr, "\tau_1"] & \\ \mathbb{H}P^n & & \mathbb{C}P^{2n+1} \arrow[ll, "\pi"'] \end{tikzcd}$$

其中 $\tau_1: Sp(n+1) \to \mathbb{C}P^{2n+1}, \boldsymbol{H} = \boldsymbol{U} + j\boldsymbol{V} \mapsto \begin{bmatrix} \boldsymbol{U}_0 \\ \boldsymbol{V}_0 \end{bmatrix}$ 为自然投影，$\boldsymbol{U} = (\boldsymbol{U}_0, \cdots, \boldsymbol{U}_n)$，$\boldsymbol{V} = (\boldsymbol{V}_0, \cdots, \boldsymbol{V}_n) \in GL(n+1;\mathbb{C})$，其中将 $Sp(n)$ 中的元素 $\boldsymbol{U} + j\boldsymbol{V}$ 看作 $SU(n)$ 中的元素$\begin{pmatrix} \boldsymbol{U} & -\bar{\boldsymbol{V}} \\ \boldsymbol{V} & \bar{\boldsymbol{U}} \end{pmatrix}$。$\tau_2: Sp(n+1) \to \mathbb{H}P^n, \boldsymbol{H} = (\boldsymbol{H}_0, \cdots, \boldsymbol{H}_n) \to [\boldsymbol{H}_0]_{\mathbb{H}}$。

定义 1.1 在上述交换图中我们称 $\pi: \mathbb{C}P^{2n+1} \to \mathbb{H}P^n, \begin{bmatrix} \boldsymbol{U}_0 \\ \boldsymbol{V}_0 \end{bmatrix} \to [\boldsymbol{U}_0 + j\boldsymbol{V}_0]_{\mathbb{H}}$ 为扭映射.

定义 1.2 扭映射 $\pi: \mathbb{C}P^{2n+1} \to \mathbb{H}P^n$ 给出$\mathbb{C}P^{2n+1}$ 的水平切空间与$\mathbb{H}P^n$ 的切空间一个自然等同，所以定义$\mathbb{C}P^{2n+1}$ 在点$[u]$ 的水平部分 $\mathrm{T}_{[u]}$ 为 $\pi([u])$ 处纤维的正交补，其中 $\mathbb{C}P^{2n+1}$ 上的度量为 Fubini-Study 度量. 若 $\Omega: N \to \mathbb{C}P^{2n+1}$ 称为水平的，Ω 的切映射的像落在 $\mathrm{T}_{[u]}$ 中。

任意的 $\boldsymbol{u} = \left[\begin{pmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_2 \end{pmatrix}\right] \in \mathbb{C}^{2n+2}$，由 Yang[8] 的结论可知

$$\mathrm{T}_{[u]} \leftrightarrow \{\boldsymbol{v} \in \boldsymbol{u}^{\perp} \mid \sigma_u(\boldsymbol{v}) = 0,$$
$$\sigma_u = -\boldsymbol{v}_2^{\mathrm{T}}\mathrm{d}\boldsymbol{u}_1 + \boldsymbol{u}_1^{\mathrm{T}}\mathrm{d}\boldsymbol{u}_2, \boldsymbol{u}_1, \boldsymbol{u}_2 \in \mathbb{C}^{n+1}\}. \quad (1)$$

2 $\mathbb{H}P^3$ 中的共形极小曲面

2.1 $\mathbb{H}P^3$ 中的共形极小曲面

命题 2.1 设 $\Omega = [\boldsymbol{\omega}] = \left[\begin{pmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_2 \end{pmatrix}\right]: N \to \mathbb{C}P^{2n+1}$ 为浸入，则 Ω 为水平的当且仅当

$$\begin{cases} \langle \boldsymbol{\omega}, \partial\boldsymbol{v} \rangle = 0 \\ \langle \boldsymbol{\omega}, \bar{\partial}\boldsymbol{v} \rangle = 0 \end{cases}，其中[\boldsymbol{v}] = \left[\begin{pmatrix} -\bar{\boldsymbol{u}}_2 \\ \bar{\boldsymbol{u}}_1 \end{pmatrix}\right].$$

证明 由式(1)知 Ω 水平$\Leftrightarrow \left\langle \begin{pmatrix} \mathrm{d}\boldsymbol{u}_1 \\ \mathrm{d}\boldsymbol{u}_2 \end{pmatrix}, \begin{pmatrix} -\bar{\boldsymbol{u}}_2 \\ \bar{\boldsymbol{u}}_1 \end{pmatrix} \right\rangle = \langle \mathrm{d}\boldsymbol{\omega}, \boldsymbol{v} \rangle = 0 \Leftrightarrow \begin{cases} \langle \partial\boldsymbol{\omega}, \boldsymbol{v} \rangle = 0 \\ \langle \bar{\partial}\boldsymbol{\omega}, \boldsymbol{v} \rangle = 0, \end{cases}$ 又由$\langle \boldsymbol{\omega}, \boldsymbol{v} \rangle = 0$，在等式两边分别作用$\partial$和$\bar{\partial}$，即可得所证。

根据 Bolton 等[3]给出的结论,有如下定义:

定义 2.1　称 $\{\Phi_0,\cdots,\Phi_n\}$ 为 $\mathbb{C}P^n$ 中的 Veronese 序列,其中 $\Phi_s:S^2\to\mathbb{C}P^n$,$\boldsymbol{u}\mapsto[(\phi_{s,0}(u),\cdots,\phi_{s,n}(u))^{\mathrm T}]$ 为曲率为 $\dfrac{4}{n+2s(n-s)}$ 的共形极小二维球面,其中 $s=0,1,\cdots,n$,u 为 S^2 上的全纯坐标。对于 $t=0,1,\cdots,n$,

$$\phi_{s,t}(u)=\frac{s!}{(1+u\bar u)^s}\sqrt{C_n^t}\,u^{t-s}\sum_k(-1)^kC_t^{s-k}C_{n-t}^k(u\bar u)^k.$$

例 1　通过上述可算得 $\mathbb{C}P^7$ 的 Veronese 序列如下:

$$\begin{cases}
\boldsymbol{\Phi}_0^{(7)}=[(1,\sqrt7z,\sqrt{21}z^2,\sqrt{35}z^3,\sqrt{35}z^4,\sqrt{21}z^5,\sqrt7z^6,z^7)^{\mathrm T}]\\
\boldsymbol{\Phi}_1^{(7)}=[(-7\bar z,\sqrt7(1-6z\bar z),\sqrt{21}(2z-5z^2\bar z),\sqrt{35}(3z^2-4z^3\bar z),\sqrt{35}(4z^3-3z^4\bar z),\\
\quad\sqrt{21}(5z^4-2z^5\bar z),\sqrt7(6z^5-z^6\bar z),7z^6)^{\mathrm T}]\\
\boldsymbol{\Phi}_2^{(7)}=[(21\bar z^2,\sqrt7(-6\bar z+15z\bar z^2),\sqrt{21}(1-10z\bar z+10z^2\bar z^2),\sqrt{35}(3z-12z^2\bar z+6z^3\bar z^2),\\
\quad\sqrt{35}(6z^2-12z^3\bar z+3z^4\bar z^2),\sqrt{21}(10z^3-10z^4\bar z+z^5\bar z^2),\sqrt7(15z^4-6z^5\bar z),21z^5)^{\mathrm T}]\\
\boldsymbol{\Phi}_3^{(7)}=[(-35\bar z^3,\sqrt7(15\bar z^2-20z\bar z^3),\sqrt{21}(-5\bar z+20z\bar z^2-10z^2\bar z^3),\sqrt{35}(1-12z\bar z-18z^2\bar z^2-4z^3\bar z^3),\\
\quad\sqrt{35}(4z-18z^2\bar z+12z^3\bar z^2-z^4\bar z^3),\sqrt{21}(10z^2-20z^3\bar z+5z^4\bar z^2),\sqrt7(20z^3-15z^4\bar z),35z^4)^{\mathrm T}]\\
\boldsymbol{\Phi}_4^{(7)}=[(35\bar z^4,\sqrt7(-20\bar z^3+15z\bar z^4),\sqrt{21}(10\bar z^2-20z\bar z^3+5z^2\bar z^4),\sqrt{35}(-4\bar z+18z\bar z^2-12z^2\bar z^3+z^3\bar z^4),\\
\quad\sqrt{35}(1-12z\bar z+18z^2\bar z^2-4z^3\bar z^3),\sqrt{21}(5z-20z^2\bar z+10z^3\bar z^2),\sqrt7(15z^2-20z^3\bar z),35z^3)^{\mathrm T}]\\
\boldsymbol{\Phi}_5^{(7)}=[(-21\bar z^5,\sqrt7(15\bar z^4-6z\bar z^5),\sqrt{21}(-10\bar z^3+10z\bar z^4-z^2\bar z^5),\sqrt{35}(6\bar z^2-12z\bar z^3+3z^2\bar z^4),\\
\quad\sqrt{35}(-3\bar z+12z\bar z^2-6z^2\bar z^3),\sqrt{21}(1-10z\bar z+10z^2\bar z^2),\sqrt7(6z-15z^2\bar z),21z^2)^{\mathrm T}]\\
\boldsymbol{\Phi}_6^{(7)}=[(7\bar z^6,\sqrt7(-6\bar z^5+z\bar z^6),\sqrt{21}(5\bar z^4-2z\bar z^5),\sqrt{35}(-4\bar z^3+3z\bar z^4),\sqrt{35}(3\bar z^2-4z\bar z^3),\\
\quad\sqrt{21}(-2\bar z+5z\bar z^2),\sqrt7(1-6z\bar z),7z)^{\mathrm T}]\\
\boldsymbol{\Phi}_7^{(7)}=[(-\bar z^7,\sqrt7\bar z^6,-\sqrt{21}\bar z^5,\sqrt{35}\bar z^4,-\sqrt{35}\bar z^3,\sqrt{21}\bar z^2,-\sqrt7\bar z,1)^{\mathrm T}]
\end{cases}$$

对应的 Gaussian 曲率分别为 4/7,4/19,4/27,4/31,4/31,4/27,4/19,4/7。

接下来给出两个引理:

引理 2.1[7]　对于水平极小曲面 $\Omega=[\boldsymbol{\omega}]:N\to\mathbb{C}P^{2n+1}$,则 $\pi\circ\Omega:N\to\mathbb{H}P^n$ 为 $\mathbb{H}P^n$ 中的共形极小曲面,其中

$$\pi:\mathbb{C}P^{2n+1}\to\mathbb{H}P^n \text{ 为扭映射。}$$

引理 2.2[3]　设 $\Phi:S^2\to\mathbb{C}P^n$ 为线性满的常曲率共形极小浸入,那么在差一个全纯等距意义下,Φ 与 $\mathbb{C}P^n$ 中的 Vernoese 序列可以看作等同。

这样根据上述两个引理,可以得到以下命题。

命题 2.2　设 $\boldsymbol{U}\in\boldsymbol{U}(2n+2)$,若 $U\cdot\Phi_s$ 为水平的,其中 $\Phi_s:S^2\to\mathbb{C}P^{2n+1}$ 为 Veronese 序列中的一个元素,则 $\pi\circ(\boldsymbol{U}\cdot\Phi_s):S^2\to\mathbb{H}P^n$ 为曲率是 $\dfrac{4}{2n+1+2s(2n+1-s)}$ 的共形极小曲面,其中 $\pi:\mathbb{C}P^{2n+1}\to\mathbb{H}P^n$ 为扭映射。

接下来利用 Veronese 序列介绍文献[7]给出的构造 $\mathbb{H}P^n$ 中常曲率极小二维球面的构造方法。

定理 2.1　设 $\boldsymbol{\Phi}=[\boldsymbol{\omega}]=[(\omega_1,\cdots,\omega_{2n+2})^{\mathrm T}]:S^2\to\mathbb{C}P^{2n+1}$ 为浸入,若存在 $\boldsymbol{U}\in\boldsymbol{U}(2n+2)$ 使得 $\boldsymbol{U}\cdot\boldsymbol{\Phi}$ 水平当且仅当

$$\sum_{p,q=1}^{2n+2}A_{pq}\omega_p\partial\omega_q=0,\ \sum_{p,q=1}^{2n+2}A_{pq}\omega_p\bar\partial\omega_q=0,\qquad(2)$$

其中 $\boldsymbol{A}=\boldsymbol{U}^{\mathrm T}\begin{pmatrix}0&\boldsymbol{I}_{n+1}\\-\boldsymbol{I}_{n+1}&0\end{pmatrix}\boldsymbol{U}=(A_{pq})$ 为反对称矩阵。

证明　由命题 2.1 知,$\boldsymbol{U}\cdot\boldsymbol{\Phi}$ 水平 $\Leftrightarrow\begin{cases}\langle\boldsymbol{U\omega},\partial\boldsymbol{v}\rangle=0\\\langle\boldsymbol{U\omega},\bar\partial\boldsymbol{v}\rangle=0\end{cases}$,其中 $\boldsymbol{v}=\begin{pmatrix}0&-\boldsymbol{I}_{n+1}\\\boldsymbol{I}_{n+1}&0\end{pmatrix}\cdot(\overline{\boldsymbol{U}}\,\overline{\boldsymbol{\omega}})$. 利用复内积运算将上述展开,可得 $\begin{cases}\partial\boldsymbol{\omega}^{\mathrm T}\boldsymbol{U}^{\mathrm T}\begin{pmatrix}0&\boldsymbol{I}_{n+1}\\-\boldsymbol{I}_{n+1}&0\end{pmatrix}\boldsymbol{U\omega}=0\\\bar\partial\boldsymbol{\omega}^{\mathrm T}\boldsymbol{U}^{\mathrm T}\begin{pmatrix}0&\boldsymbol{I}_{n+1}\\-\boldsymbol{I}_{n+1}&0\end{pmatrix}\boldsymbol{U\omega}=0\end{cases}$,记 $\boldsymbol{A}=\boldsymbol{U}^{\mathrm T}\begin{pmatrix}0&\boldsymbol{I}_{n+1}\\-\boldsymbol{I}_{n+1}&0\end{pmatrix}\boldsymbol{U}=(A_{pq})$,则第一个等式展开

$$(\partial\omega_1,\cdots,\partial\omega_{2n+2})\begin{pmatrix}A_{11}&\cdots&A_{1\,2n+2}\\\vdots&&\vdots\\A_{2n+2\,1}&\cdots&A_{2n+2\,2n+2}\end{pmatrix}\begin{pmatrix}\omega_1\\\vdots\\\omega_{2n+2}\end{pmatrix}=$$

$$\left(\sum_{p=1}^{2n+1} A_{p\,1}\partial\omega_p, \cdots, \sum_{p=1}^{2n+1} A_{p\,2n+2}\partial\omega_p\right)\begin{pmatrix}\omega_1\\ \vdots\\ \omega_{2n+2}\end{pmatrix} = 0.$$

同理,将第 2 个等式展开,(2)式得证。

2.2 $\mathbb{H}P^3$ 中的共形极小二维球面

接下来利用上述定理,以 $n=3$ 为例,构造 $\mathbb{H}P^3$ 中的共形极小二维球面例子。而根据命题 2.2,若 $\Phi_s=\Phi_s^{(7)}$ 为例 1 中给出的 $\mathbb{C}P^7$ 中的 Veronese 序列中一个元素,能找到 $\boldsymbol{U}\in\boldsymbol{U}(8)$ 使得 $\Phi_s\cdot\boldsymbol{U}$ 为水平的,则 $\pi\circ(\boldsymbol{U}\cdot\Phi_s): S^2\to\mathbb{H}P^3$ 为曲率是 $\dfrac{4}{7+2s(7-s)}$ 的共形极小二维球面,而使得 $\Phi_s\cdot\boldsymbol{U}$ 为水平的条件已由定理2.1 给出。

情形 1:$\boldsymbol{\Phi}=\boldsymbol{\Phi}_0^{(7)}$。

由例 1 知,$\boldsymbol{\Phi}_0^{(7)}=[(1,\sqrt{7}z,\sqrt{21}z^2,\sqrt{35}z^3,\sqrt{35}z^4,\sqrt{21}z^5,\sqrt{7}z^6,z^7)^{\mathrm{T}}]$,则式(2)可展开如下:

$$\begin{aligned}\sum_{p,q=1}^{8} A_{pq}\omega_p\partial\omega_q ={}& \sqrt{7}A_{12}+2\sqrt{21}A_{13}z+(3\sqrt{35}A_{14}+7\sqrt{3}A_{23})z^2+(4\sqrt{35}A_{15}+14\sqrt{5}A_{24})z^3+(5\sqrt{21}A_{16}+\\ &21\sqrt{5}A_{25}+7\sqrt{15}A_{34})z^4+(6\sqrt{7}A_{17}+28\sqrt{3}A_{26}+14\sqrt{15}A_{35})z^5+(7A_{18}+35A_{27}+63A_{36}+\\ &35A_{45})z^6+(6\sqrt{7}A_{28}+28\sqrt{3}A_{37}+14\sqrt{15}A_{46})z^7+(5\sqrt{21}A_{38}+21\sqrt{5}A_{47}+7\sqrt{15}A_{56})z^8+\\ &(4\sqrt{35}A_{48}+14\sqrt{5}A_{57})z^9+(3\sqrt{35}A_{58}+7\sqrt{3}A_{67})z^{10}=0,\end{aligned}$$

$$\sum_{p,q=1}^{8} A_{pq}\omega_p\bar{\partial}\omega_q = 0.$$

由于 $\boldsymbol{A}$ 为一个反对称矩阵,所以可算得

$$\begin{cases}A_{12}=A_{13}=A_{68}=A_{78}=0,\\ 3\sqrt{35}A_{14}+7\sqrt{3}A_{23}=0,\\ 4\sqrt{35}A_{15}+14\sqrt{5}A_{24}=0,\\ 5\sqrt{21}A_{16}+21\sqrt{5}A_{25}+7\sqrt{15}A_{34}=0,\\ 6\sqrt{7}A_{17}+28\sqrt{3}A_{26}+14\sqrt{15}A_{35}=0,\\ 7A_{18}+35A_{27}+63A_{36}+35A_{45}=0,\\ 6\sqrt{7}A_{28}+28\sqrt{3}A_{37}+14\sqrt{15}A_{46}=0,\\ 5\sqrt{21}A_{38}+21\sqrt{5}A_{47}+7\sqrt{15}A_{56}=0,\\ 4\sqrt{35}A_{48}+14\sqrt{5}A_{57}=0,\\ 3\sqrt{35}A_{58}+7\sqrt{3}A_{67}=0.\end{cases}$$

所以,由定理 2.1,$\boldsymbol{U}\cdot\boldsymbol{\Phi}_0^{(7)}$ 水平当且仅当 $\boldsymbol{U}$ 满足

$\boldsymbol{A}=\boldsymbol{U}^{\mathrm{T}}\begin{pmatrix}0 & \boldsymbol{I}_4\\ -\boldsymbol{I}_4 & 0\end{pmatrix}\boldsymbol{U}$,其中 $\boldsymbol{A}=(\boldsymbol{A}_1\boldsymbol{A}_2)$,

$$\boldsymbol{A}_1=\begin{pmatrix}0 & 0 & 0 & A_{14}\\ 0 & 0 & -\dfrac{\sqrt{105}}{7}A_{14} & -\dfrac{2\sqrt{7}}{7}A_{15}\\ 0 & \dfrac{\sqrt{105}}{7}A_{14} & 0 & -\dfrac{\sqrt{35}}{7}A_{16}-\dfrac{\sqrt{3}}{9}A_{25}\\ -A_{14} & \dfrac{2\sqrt{7}}{7}A_{15} & \dfrac{\sqrt{35}}{7}A_{16}+\dfrac{\sqrt{3}}{9}A_{25} & 0\\ -A_{15} & -A_{25} & \dfrac{\sqrt{105}}{35}A_{17}+\dfrac{2\sqrt{5}}{5}A_{26} & \dfrac{1}{5}A_{18}+A_{27}+\dfrac{9}{5}A_{36}\\ -A_{16} & -A_{26} & -A_{36} & \dfrac{\sqrt{105}}{35}A_{28}+\dfrac{2\sqrt{5}}{5}A_{37}\\ -A_{17} & -A_{27} & -A_{37} & -A_{47}\\ -A_{18} & -A_{28} & -A_{38} & -A_{48}\end{pmatrix}$$

$$
\boldsymbol{A}_2 = \begin{pmatrix}
A_{15} & A_{16} & A_{17} & A_{18} \\
A_{25} & A_{26} & A_{27} & A_{28} \\
-\frac{\sqrt{105}}{35}A_{17} - \frac{2\sqrt{5}}{5}A_{26} & A_{36} & A_{37} & A_{38} \\
-\frac{1}{5}A_{18} - A_{27} - \frac{9}{5}A_{36} & -\frac{\sqrt{105}}{35}A_{28} - \frac{2\sqrt{5}}{5}A_{37} & A_{47} & A_{48} \\
0 & -\frac{\sqrt{35}}{7}A_{38} - \sqrt{3}A_{47} & -\frac{2\sqrt{7}}{7}A_{48} & A_{58} \\
\frac{\sqrt{35}}{7}A_{38} + \sqrt{3}A_{47} & 0 & -\frac{\sqrt{105}}{7}A_{58} & 0 \\
\frac{2\sqrt{7}}{7}A_{48} & \frac{\sqrt{105}}{7}A_{58} & 0 & 0 \\
-A_{58} & 0 & 0 & 0
\end{pmatrix}
$$

情形 2:$\boldsymbol{\Phi}=\boldsymbol{\Phi}_1^{(7)}$。

由例 1 知,$\boldsymbol{\Phi}_1^{(7)} = [(-7\bar{z}, \sqrt{7}(1-6z\bar{z}), \sqrt{21}(2z-5z^2\bar{z}), \sqrt{35}(3z^2-4z^3\bar{z}), \sqrt{35}(4z^3-3z^4\bar{z}), \sqrt{21}(5z^4-2z^5\bar{z}), \sqrt{7}(6z^5-z^6\bar{z}), 7z^6)^{\mathrm{T}}]$,则式(2)可展开如下:

$$
\begin{aligned}
\sum_{p,q=1}^{8} A_{pq}\omega_p\partial\omega_q = {} & 14\sqrt{3}A_{23} + 70\sqrt{21}A_{13}z\bar{z}^2 + 42\sqrt{5}A_{24}z + 42\sqrt{7}A_{12}\bar{z}^2 - (42\sqrt{35}A_{14} + 70\sqrt{3}A_{23})z\bar{z} + \\
& (84\sqrt{5}A_{25} + 42\sqrt{15}A_{34})z^2 - 14\sqrt{21}A_{13}\bar{z} + (140\sqrt{3}A_{26} + 112\sqrt{15}A_{35})z^3 + (210A_{27} + \\
& 630A_{36} + 420A_{45})z^4 - (84\sqrt{35}A_{15} + 210\sqrt{5}A_{24})z^2\bar{z} - (140\sqrt{21}A_{16} + 420\sqrt{5}A_{25} + \\
& 112\sqrt{15}A_{34})z^3\bar{z} + (84\sqrt{35}A_{14} + 210\sqrt{3}A_{23})z^2\bar{z}^2 + (84\sqrt{35}A_{15} + 336\sqrt{5}A_{24})z^3\bar{z}^2 - \\
& (210\sqrt{7}A_{17} + 700\sqrt{3}A_{26} + 266\sqrt{15}A_{35})z^4\bar{z} + (42\sqrt{7}A_{28} + 336\sqrt{3}A_{37} + 210\sqrt{15}A_{46})z^5 + \\
& (70\sqrt{21}A_{16} + 378\sqrt{5}A_{25} + 140\sqrt{15}A_{34})z^4\bar{z}^2 - (294A_{18} + 1050A_{27} + 1386A_{36} + \\
& 630A_{45})z^5\bar{z} + (70\sqrt{21}A_{38} + 378\sqrt{5}A_{47} + 140\sqrt{15}A_{56})z^6 + (42\sqrt{7}A_{17} + 336\sqrt{3}A_{26} + \\
& 210\sqrt{15}A_{35})z^5\bar{z}^2 - (140\sqrt{21}A_{38} + 420\sqrt{5}A_{47} + 112\sqrt{15}A_{56})z^7\bar{z} + (84\sqrt{35}A_{48} + \\
& 336\sqrt{5}A_{57})z^7 + (210A_{27} + 630A_{36} + 420A_{45})z^6\bar{z}^2 - (210\sqrt{7}A_{28} + 700\sqrt{3}A_{37} + \\
& 266\sqrt{15}A_{46})z^6\bar{z} + (84\sqrt{35}A_{58} + 210\sqrt{3}A_{67})z^8 + (140\sqrt{3}A_{37} + 112\sqrt{15}A_{46})z^7\bar{z}^2 - \\
& (84\sqrt{35}A_{48} + 210\sqrt{5}A_{57})z^8\bar{z} + 70\sqrt{21}A_{68}z^9 + (84\sqrt{5}A_{47} + 42\sqrt{15}A_{56})z^8\bar{z}^2 - \\
& (42\sqrt{35}A_{58} + 70\sqrt{3}A_{67})z^9\bar{z} + 42\sqrt{7}A_{78}z^{10} + 42\sqrt{5}A_{57}z^9\bar{z}^2 - 14\sqrt{21}A_{68}z^{10}\bar{z} + \\
& 14\sqrt{3}A_{67}z^{10}\bar{z}^2 = 0,
\end{aligned}
$$

$$
\begin{aligned}
\sum_{p,q=1}^{8} A_{pq}\omega_p\bar{\partial}\omega_q = {} & -7\sqrt{7}A_{21} - 14\sqrt{21}A_{31}z - (49\sqrt{3}A_{32} + 21\sqrt{35}A_{41})z^2 - (98\sqrt{5}A_{42} + 28\sqrt{35}A_{51})z^3 - \\
& (49\sqrt{15}A_{43} + 147\sqrt{5}A_{52} + 35\sqrt{21}A_{61})z^4 - (98\sqrt{15}A_{53} + 196\sqrt{3}A_{62} + 42\sqrt{7}A_{71})z^5 - \\
& (245A_{54} + 441A_{63} + 245A_{72} + 49A_{81})z^6 - (98\sqrt{15}A_{64} + 196\sqrt{3}A_{73} + 42\sqrt{7}A_{82})z^7 - \\
& (49\sqrt{15}A_{65} + 147\sqrt{5}A_{74} + 35\sqrt{21}A_{83})z^8 - (98\sqrt{5}A_{75} + 28\sqrt{35}A_{84})z^9 - \\
& (49\sqrt{3}A_{76} + 21\sqrt{35}A_{85})z^{10} - 14\sqrt{21}A_{86}z^{11} - 7\sqrt{7}A_{87}z^{12} = 0.
\end{aligned}
$$

由于 $\boldsymbol{A}$ 为一个反对称矩阵,所以可算得

$$
\begin{cases}
A_{23}=A_{13}=A_{24}=A_{12}=A_{78}=0\\
A_{86}=A_{48}=A_{58}=A_{75}=A_{76}=0\\
2\sqrt{5}A_{25}+\sqrt{15}A_{34}=0\\
5\sqrt{21}A_{16}+15\sqrt{5}A_{25}+4\sqrt{15}A_{34}=0\\
5\sqrt{21}A_{16}+27\sqrt{5}A_{25}+10\sqrt{15}A_{34}=0\\
5\sqrt{3}A_{26}+4\sqrt{15}A_{35}=0\\
15\sqrt{7}A_{17}+50\sqrt{3}A_{26}+19\sqrt{15}A_{35}=0\\
\sqrt{7}A_{17}+8\sqrt{3}A_{26}+15\sqrt{15}A_{35}=0\\
A_{27}+3A_{36}+2A_{45}=0\\
294A_{18}+1050A_{27}+1386A_{36}+630A_{45}=0\\
A_{18}+5A_{27}+9A_{36}+5A_{45}=0\\
5\sqrt{21}A_{38}+21\sqrt{5}A_{47}+7\sqrt{15}A_{56}=0\\
5\sqrt{21}A_{38}+21\sqrt{5}A_{47}+10\sqrt{15}A_{56}=0\\
5\sqrt{21}A_{38}+15\sqrt{5}A_{47}+4\sqrt{15}A_{56}=0\\
3\sqrt{7}A_{28}+14\sqrt{3}A_{37}+7\sqrt{15}A_{46}=0\\
\sqrt{7}A_{28}+8\sqrt{3}A_{37}+5\sqrt{15}A_{46}=0\\
105\sqrt{7}A_{28}+350\sqrt{3}A_{37}+133\sqrt{15}A_{46}=0
\end{cases}
$$

所以,由定理 2.1, $\boldsymbol{U}\cdot\boldsymbol{\Phi}_1^{(7)}$ 水平当且仅当 $\boldsymbol{U}$ 满足

$\boldsymbol{A}=\boldsymbol{U}^{\mathrm{T}}\begin{pmatrix}0 & \boldsymbol{I}_4\\ -\boldsymbol{I}_4 & 0\end{pmatrix}\boldsymbol{U}$,其中 $\boldsymbol{A}=(\boldsymbol{A}_1\boldsymbol{A}_2)$,

$$
\boldsymbol{A}_1=\begin{pmatrix}
0 & 0 & 0 & 0\\
0 & 0 & 0 & 0\\
0 & 0 & 0 & \frac{2\sqrt{35}}{7}A_{16}\\
0 & 0 & -\frac{2\sqrt{35}}{7}A_{16} & 0\\
0 & \frac{\sqrt{105}}{7}A_{16} & -\frac{\sqrt{105}}{21}A_{17} & -A_{18}-2A_{27}\\
-A_{16} & \frac{4\sqrt{21}}{21}A_{17} & \frac{5}{3}A_{27}+\frac{2}{3}A_{18} & -\frac{\sqrt{105}}{21}A_{28}\\
-A_{17} & -A_{27} & \frac{4\sqrt{21}}{21}A_{28} & \frac{\sqrt{105}}{7}A_{38}\\
-A_{18} & -A_{28} & -A_{38} & 0
\end{pmatrix},
$$

$$
\boldsymbol{A}_2=\begin{pmatrix}
0 & A_{16} & A_{17} & A_{18}\\
-\frac{\sqrt{105}}{7}A_{16} & -\frac{4\sqrt{21}}{21}A_{17} & A_{27} & A_{28}\\
\frac{\sqrt{105}}{21}A_{17} & -\frac{5}{3}A_{27}-\frac{2}{3}A_{18} & -\frac{4\sqrt{21}}{21}A_{28} & A_{38}\\
A_{18}+2A_{27} & \frac{\sqrt{105}}{21}A_{28} & -\frac{\sqrt{105}}{7}A_{38} & 0\\
0 & \frac{2\sqrt{35}}{7}A_{38} & 0 & 0\\
-\frac{2\sqrt{35}}{7}A_{38} & 0 & 0 & 0\\
0 & 0 & 0 & 0\\
0 & 0 & 0 & 0
\end{pmatrix}.
$$

情形 3:$\boldsymbol{\Phi}=\boldsymbol{\Phi}_2^{(7)}$。

由例 1 知, $\boldsymbol{\Phi}_2^{(7)}=[(21\bar{z}^2,\sqrt{7}(-6\bar{z}+15z\bar{z}^2),\sqrt{21}(1-10z\bar{z}+10z^2\bar{z}^2),\sqrt{35}(3z-12z^2\bar{z}+6z^3\bar{z}^2),\sqrt{35}(6z^2-12z^3\bar{z}+3z^4\bar{z}^2),\sqrt{21}(10z^3-10z^4\bar{z}+z^5\bar{z}^2),\sqrt{7}(15z^4-6z^5\bar{z}),21z^5)^{\mathrm{T}}]$, 则由式(2)得

$$\sum_{p,q=1}^{8}A_{pq}\omega_p\partial\omega_q=0,\ \sum_{p,q=1}^{8}A_{pq}\omega_p\bar{\partial}\omega_q=0.$$

由于 $\boldsymbol{A}$ 为一个反对称矩阵,所以可算得

$$
\begin{cases}
A_{12}=A_{13}=A_{14}=A_{15}=A_{16}=A_{17}=0\\
A_{23}=A_{24}=A_{25}=A_{26}=A_{28}=0\\
A_{34}=A_{35}=A_{37}=A_{38}=0\\
A_{46}=A_{47}=A_{48}=A_{56}=A_{57}=A_{58}=0\\
A_{67}=A_{68}=A_{78}=0\\
7A_{18}+19A_{27}+27A_{36}+15A_{45}=0\\
A_{27}+2A_{36}+A_{45}=0\\
7A_{18}+25A_{27}+33A_{36}+15A_{45}=0\\
A_{36}+A_{45}=0
\end{cases}
$$

所以,由定理 2.1, $\boldsymbol{U}\cdot\boldsymbol{\Phi}_2^{(7)}$ 水平当且仅当 $\boldsymbol{U}$ 满足

$\boldsymbol{A}=\boldsymbol{U}^{\mathrm{T}}\begin{pmatrix}0 & \boldsymbol{I}_4\\ -\boldsymbol{I}_4 & 0\end{pmatrix}\boldsymbol{U}$, 其中

$$
\boldsymbol{A}=\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{18}\\
0 & 0 & 0 & 0 & 0 & 0 & -A_{18} & 0\\
0 & 0 & 0 & 0 & 0 & A_{18} & 0 & 0\\
0 & 0 & 0 & 0 & -A_{18} & 0 & 0 & 0\\
0 & 0 & 0 & A_{18} & 0 & 0 & 0 & 0\\
0 & 0 & -A_{18} & 0 & 0 & 0 & 0 & 0\\
0 & A_{18} & 0 & 0 & 0 & 0 & 0 & 0\\
-A_{18} & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
$$

情形 4:$\boldsymbol{\Phi}=\boldsymbol{\Phi}_3^{(7)}$。

由例 1 知, $\boldsymbol{\Phi}_3^{(7)}=[(-35\bar{z}^3,\sqrt{7}(15\bar{z}^2-20z\bar{z}^3),\sqrt{21}(-5\bar{z}+20z\bar{z}^2-10z^2\bar{z}^3),\sqrt{35}(1-12z\bar{z}-18z^2\bar{z}^2-4z^3\bar{z}^3),\sqrt{35}(4z-18z^2\bar{z}+12z^3\bar{z}^2-z^4\bar{z}^3),\sqrt{21}(10z^2-20z^3\bar{z}+5z^4\bar{z}^2),\sqrt{7}(20z^3-15z^4\bar{z}),35z^4)^{\mathrm{T}}]$, 则由式(2)得

$$\sum_{p,q=1}^{8}A_{pq}\omega_p\partial\omega_q=0,\ \sum_{p,q=1}^{8}A_{pq}\omega_p\bar{\partial}\omega_q=0.$$

由于 $\boldsymbol{A}$ 为一个反对称矩阵,所以可算得

$$\begin{cases} A_{12}=A_{13}=A_{15}=A_{16}=A_{17}=A_{18}=0 \\ A_{24}=A_{25}=A_{26}=A_{27}=A_{28}=0 \\ A_{34}=A_{35}=A_{36}=A_{37}=A_{38}=0 \\ A_{45}=A_{46}=A_{47}=A_{48}=A_{56}=A_{57}=0 \\ A_{68}=A_{78}=0 \\ 42\sqrt{35}A_{14}+140\sqrt{3}A_{23}=0 \\ 126\sqrt{35}A_{14}+210\sqrt{3}A_{23}=0 \\ 42\sqrt{35}A_{58}+140\sqrt{3}A_{67}=0 \\ 126\sqrt{35}A_{58}+210\sqrt{3}A_{67}=0 \end{cases}$$

所以,由定理 2.1, $\boldsymbol{U}\cdot\boldsymbol{\Phi}_3^{(7)}$ 水平当且仅当 $\boldsymbol{U}$ 满足 $\boldsymbol{A}=\boldsymbol{U}^{\mathrm{T}}\begin{pmatrix}0 & \boldsymbol{I}_4\\ -\boldsymbol{I}_4 & 0\end{pmatrix}\boldsymbol{U}$,其中 $\boldsymbol{A}=0$.

情形 5:$\boldsymbol{\Phi}=\boldsymbol{\Phi}_4^{(7)}$。

由例 1 知,$\boldsymbol{\Phi}_4^{(7)}=[(35\bar{z}^4,\sqrt{7}(-20\bar{z}^3+15z\bar{z}^4),\sqrt{21}(10\bar{z}^2-20z\bar{z}^3+5z^2\bar{z}^4),\sqrt{35}(-4\bar{z}+18z\bar{z}^2-12z^2\bar{z}^3+z^3\bar{z}^4),\sqrt{35}(1-12z\bar{z}+18z^2\bar{z}^2-4z^3\bar{z}^3),\sqrt{21}(5z-20z^2\bar{z}+10z^3\bar{z}^2),\sqrt{7}(15z^2-20z^3\bar{z}),35z^3)^{\mathrm{T}}]$,则由式(2)得

$$\sum_{p,q=1}^{8}A_{pq}\omega_p\partial\omega_q=0,\ \sum_{p,q=1}^{8}A_{pq}\omega_p\bar{\partial}\omega_q=0.$$

由于 $\boldsymbol{A}$ 为一个反对称矩阵,所以可算得

$$\begin{cases} A_{12}=A_{13}=A_{15}=A_{16}=A_{17}=A_{18}=0 \\ A_{24}=A_{25}=A_{26}=A_{27}=A_{28}=0 \\ A_{34}=A_{35}=A_{36}=A_{37}=A_{38}=0 \\ A_{45}=A_{46}=A_{47}=A_{48}=A_{56}=A_{57}=0 \\ A_{68}=A_{78}=0 \\ 42\sqrt{35}A_{14}+140\sqrt{3}A_{23}=0 \\ 126\sqrt{35}A_{14}+210\sqrt{3}A_{23}=0 \\ 42\sqrt{35}A_{58}+140\sqrt{3}A_{67}=0 \\ 126\sqrt{35}A_{58}+210\sqrt{3}A_{67}=0 \end{cases}$$

所以,由定理 2.1, $\boldsymbol{U}\cdot\boldsymbol{\Phi}_4^{(7)}$ 水平当且仅当 $\boldsymbol{U}$ 满足 $\boldsymbol{A}=\boldsymbol{U}^{\mathrm{T}}\begin{pmatrix}0 & \boldsymbol{I}_4\\ -\boldsymbol{I}_4 & 0\end{pmatrix}\boldsymbol{U}$,其中 $\boldsymbol{A}=0$。

情形 6:$\boldsymbol{\Phi}=\boldsymbol{\Phi}_5^{(7)}$。

由例 1 知,$\boldsymbol{\Phi}_5^{(7)}=[(-21\bar{z}^5,\sqrt{7}(15\bar{z}^4-6z\bar{z}^5),\sqrt{21}(-10\bar{z}^3+10z\bar{z}^4-z^2\bar{z}^5),\sqrt{35}(6\bar{z}^2-12z\bar{z}^3+3z^2\bar{z}^4),\sqrt{35}(-3\bar{z}+12z\bar{z}^2-6z^2\bar{z}^3),\sqrt{21}(1-10z\bar{z}+10z^2\bar{z}^2),\sqrt{7}(6z-15z^2\bar{z}),21z^2)^{\mathrm{T}}]$,则(2)式得

$$\sum_{p,q=1}^{8}A_{pq}\omega_p\partial\omega_q=0,\ \sum_{p,q=1}^{8}A_{pq}\omega_p\bar{\partial}\omega_q=0.$$

由于 $\boldsymbol{A}$ 为一个反对称矩阵,所以由定理 2.1, $\boldsymbol{U}\cdot\boldsymbol{\Phi}_5^{(7)}$ 水平当且仅当 $\boldsymbol{U}$ 满足 $\boldsymbol{A}=\boldsymbol{U}^{\mathrm{T}}\begin{pmatrix}0 & \boldsymbol{I}_4\\ -\boldsymbol{I}_4 & 0\end{pmatrix}\boldsymbol{U}$, 其中

$$\boldsymbol{A}=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{18} \\ 0 & 0 & 0 & 0 & 0 & 0 & -A_{18} & 0 \\ 0 & 0 & 0 & 0 & 0 & A_{18} & 0 & 0 \\ 0 & 0 & 0 & 0 & -A_{18} & 0 & 0 & 0 \\ 0 & 0 & 0 & A_{18} & 0 & 0 & 0 & 0 \\ 0 & 0 & -A_{18} & 0 & 0 & 0 & 0 & 0 \\ 0 & A_{18} & 0 & 0 & 0 & 0 & 0 & 0 \\ -A_{18} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

情形 7:$\boldsymbol{\Phi}=\boldsymbol{\Phi}_6^{(7)}$。

由例 1 知,$\boldsymbol{\Phi}_6^{(7)}=[(7\bar{z}^6,\sqrt{7}(-6\bar{z}^5+z\bar{z}^6),\sqrt{21}(5\bar{z}^4-2z\bar{z}^5),\sqrt{35}(-4\bar{z}^3+3z\bar{z}^4),\sqrt{35}(3\bar{z}^2-4z\bar{z}^3),\sqrt{21}(-2\bar{z}+5z\bar{z}^2),\sqrt{7}(1-6z\bar{z}),7z)^{\mathrm{T}}]$,则由式(2)得

$$\sum_{p,q=1}^{8}A_{pq}\omega_p\partial\omega_q=0,\ \sum_{p,q=1}^{8}A_{pq}\omega_p\bar{\partial}\omega_q=0.$$

由于 $\boldsymbol{A}$ 为一个反对称矩阵,所以由定理 2.1, $\boldsymbol{U}\cdot\boldsymbol{\Phi}_6^{(7)}$ 水平当且仅当 $\boldsymbol{U}$ 满足:$\boldsymbol{A}=\boldsymbol{U}^{\mathrm{T}}\begin{pmatrix}0 & \boldsymbol{I}_4\\ -\boldsymbol{I}_4 & 0\end{pmatrix}\boldsymbol{U}$, 其中 $\boldsymbol{A}=(\boldsymbol{A}_1\boldsymbol{A}_2)$,

$$\boldsymbol{A}_1=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{2\sqrt{35}}{7}A_{16} \\ 0 & 0 & -\frac{2\sqrt{35}}{7}A_{16} & 0 \\ 0 & \frac{\sqrt{105}}{7}A_{16} & -\frac{\sqrt{105}}{21}A_{17} & -A_{18}-2A_{27} \\ -A_{16} & \frac{4\sqrt{21}}{21}A_{17} & \frac{5}{3}A_{27}+\frac{2}{3}A_{18} & -\frac{\sqrt{105}}{21}A_{28} \\ -A_{17} & -A_{27} & \frac{4\sqrt{21}}{21}A_{28} & \frac{\sqrt{105}}{7}A_{38} \\ -A_{18} & -A_{28} & -A_{38} & 0 \end{pmatrix},$$

$$A_2=\begin{pmatrix} 0 & A_{16} & A_{17} & A_{18} \\ -\frac{\sqrt{105}}{7}A_{16} & -\frac{4\sqrt{21}}{21}A_{17} & A_{27} & A_{28} \\ \frac{\sqrt{105}}{21}A_{17} & -\frac{5}{3}A_{27}-\frac{2}{3}A_{18} & -\frac{4\sqrt{21}}{21}A_{28} & A_{38} \\ A_{18}+2A_{27} & \frac{\sqrt{105}}{21}A_{28} & -\frac{\sqrt{105}}{7}A_{38} & 0 \\ 0 & \frac{2\sqrt{35}}{7}A_{38} & 0 & 0 \\ -\frac{2\sqrt{35}}{7}A_{38} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

情形 8：$\boldsymbol{\Phi}=\boldsymbol{\Phi}_7^{(7)}$。

由例 1 知，$\boldsymbol{\Phi}_7^{(7)}=[(-\bar{z}^7,\sqrt{7}\,\bar{z}^6,-\sqrt{21}\,\bar{z}^5,\sqrt{35}\,\bar{z}^4,-\sqrt{35}\,\bar{z}^3,\sqrt{21}\,\bar{z}^2,-\sqrt{7}\,\bar{z},1)^{\mathrm{T}}]$，则由式(2)得

$$\sum_{p,q=1}^{8}A_{pq}\omega_p\partial\omega_q=0,\ \sum_{p,q=1}^{8}A_{pq}\omega_p\bar{\partial}\omega_q=0.$$

由于 $\boldsymbol{A}$ 为一个反对称矩阵，所以由定理 2.1，$\boldsymbol{U}\cdot\boldsymbol{\Phi}_7^{(7)}$ 水平当且仅当 $\boldsymbol{U}$ 满足 $\boldsymbol{A}=\boldsymbol{U}^{\mathrm{T}}\begin{pmatrix}0 & \boldsymbol{I}_4\\ -\boldsymbol{I}_4 & 0\end{pmatrix}\boldsymbol{U}$，其中 $\boldsymbol{A}=(\boldsymbol{A}_1\boldsymbol{A}_2)$，

$$A_1=\begin{pmatrix} 0 & 0 & 0 & A_{14} \\ 0 & 0 & -\frac{\sqrt{105}}{7}A_{14} & -\frac{2\sqrt{7}}{7}A_{15} \\ 0 & \frac{\sqrt{105}}{7}A_{14} & 0 & -\frac{\sqrt{35}}{7}A_{16}-\frac{\sqrt{3}}{9}A_{25} \\ -A_{14} & \frac{2\sqrt{7}}{7}A_{15} & \frac{\sqrt{35}}{7}A_{16}+\frac{\sqrt{3}}{9}A_{25} & 0 \\ -A_{15} & -A_{25} & \frac{\sqrt{105}}{35}A_{17}+\frac{2\sqrt{5}}{5}A_{26} & \frac{1}{5}A_{18}+A_{27}+\frac{9}{5}A_{36} \\ -A_{16} & -A_{26} & -A_{36} & \frac{\sqrt{105}}{35}A_{28}+\frac{2\sqrt{5}}{5}A_{37} \\ -A_{17} & -A_{27} & -A_{37} & -A_{47} \\ -A_{18} & -A_{28} & -A_{38} & -A_{48} \end{pmatrix},$$

$$A_2=\begin{pmatrix} A_{15} & A_{16} & A_{17} & A_{18} \\ A_{25} & A_{26} & A_{27} & A_{28} \\ -\frac{\sqrt{105}}{35}A_{17}-\frac{2\sqrt{5}}{5}A_{26} & A_{36} & A_{37} & A_{38} \\ -\frac{1}{5}A_{18}-A_{27}-\frac{9}{5}A_{36} & -\frac{\sqrt{105}}{35}A_{28}-\frac{2\sqrt{5}}{5}A_{37} & A_{47} & A_{48} \\ 0 & -\frac{\sqrt{35}}{7}A_{38}-\sqrt{3}A_{47} & -\frac{2\sqrt{7}}{7}A_{48} & A_{58} \\ \frac{\sqrt{35}}{7}A_{38}+\sqrt{3}A_{47} & 0 & -\frac{\sqrt{105}}{7}A_{58} & 0 \\ \frac{2\sqrt{7}}{7}A_{48} & \frac{\sqrt{105}}{7}A_{58} & 0 & 0 \\ -A_{58} & 0 & 0 & 0 \end{pmatrix}.$$

定理 2.2 设 $\Phi:S^2\to\mathbb{C}P^7$ 为 $\mathbb{C}P^7$ 中 Veronese 序列中的一个元素，若存在 $\boldsymbol{U}\in\boldsymbol{U}(8)$ 使得 $\boldsymbol{U}\cdot\boldsymbol{\Phi}$ 为水平当且仅当 $\boldsymbol{U}$ 满足

$$\boldsymbol{A}=\boldsymbol{U}^{\mathrm{T}}\begin{pmatrix}0 & \boldsymbol{I}_4\\ -\boldsymbol{I}_4 & 0\end{pmatrix}\boldsymbol{U},$$

1)当 $\boldsymbol{\Phi}=\boldsymbol{\Phi}_0^{(7)}$ 或 $\boldsymbol{\Phi}=\boldsymbol{\Phi}_7^{(7)}$,此时 $\boldsymbol{A}=(\boldsymbol{A}_1\boldsymbol{A}_2)$ 表达式如下:

$$\boldsymbol{A}_1=\begin{pmatrix} 0 & 0 & 0 & A_{14} \\ 0 & 0 & -\frac{\sqrt{105}}{7}A_{14} & -\frac{2\sqrt{7}}{7}A_{15} \\ 0 & \frac{\sqrt{105}}{4}A_{14} & 0 & -\frac{\sqrt{35}}{7}A_{16}-\frac{\sqrt{3}}{9}A_{25} \\ -A_{14} & \frac{2\sqrt{7}}{7}A_{15} & \frac{\sqrt{35}}{7}A_{16}+\frac{\sqrt{3}}{9}A_{25} & 0 \\ -A_{15} & -A_{25} & \frac{\sqrt{105}}{35}A_{17}+\frac{2\sqrt{5}}{5}A_{26} & \frac{1}{5}A_{18}+A_{27}+\frac{9}{5}A_{36} \\ -A_{16} & -A_{26} & -A_{36} & \frac{\sqrt{105}}{35}A_{28}+\frac{2\sqrt{5}}{5}A_{37} \\ -A_{17} & -A_{27} & -A_{37} & -A_{47} \\ -A_{18} & -A_{28} & -A_{38} & -A_{48} \end{pmatrix},$$

$$\boldsymbol{A}_2=\begin{pmatrix} A_{15} & A_{16} & A_{17} & A_{18} \\ A_{25} & A_{26} & A_{27} & A_{28} \\ -\frac{\sqrt{105}}{35}A_{17}-\frac{2\sqrt{5}}{5}A_{26} & A_{36} & A_{37} & A_{38} \\ -\frac{1}{5}A_{18}-A_{27}-\frac{9}{5}A_{36} & -\frac{\sqrt{105}}{35}A_{28}-\frac{2\sqrt{5}}{5}A_{37} & A_{47} & A_{48} \\ 0 & -\frac{\sqrt{35}}{7}A_{38}-\sqrt{3}A_{47} & -\frac{2\sqrt{7}}{7}A_{48} & A_{58} \\ \frac{\sqrt{35}}{7}A_{38}+\sqrt{3}A_{47} & 0 & -\frac{\sqrt{105}}{7}A_{58} & 0 \\ \frac{2\sqrt{7}}{7}A_{48} & \frac{\sqrt{105}}{7}A_{58} & 0 & 0 \\ -A_{58} & 0 & 0 & 0 \end{pmatrix}.$$

2)当 $\boldsymbol{\Phi}=\boldsymbol{\Phi}_1^{(7)}$ 或 $\boldsymbol{\Phi}=\boldsymbol{\Phi}_6^{(7)}$,此时 $\boldsymbol{A}=(\boldsymbol{A}_1\boldsymbol{A}_2)$ 表达式如下:

$$\boldsymbol{A}_1=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{2\sqrt{35}}{7}A_{16} \\ 0 & 0 & -\frac{2\sqrt{35}}{7}A_{16} & 0 \\ 0 & \frac{\sqrt{105}}{7}A_{16} & -\frac{\sqrt{105}}{21}A_{17} & -A_{18}-2A_{27} \\ -A_{16} & \frac{4\sqrt{21}}{21}A_{17} & \frac{5}{3}A_{27}+\frac{2}{3}A_{18} & -\frac{\sqrt{105}}{21}A_{28} \\ -A_{17} & -A_{27} & \frac{4\sqrt{21}}{21}A_{28} & \frac{\sqrt{105}}{7}A_{38} \\ -A_{18} & -A_{28} & -A_{38} & 0 \end{pmatrix},\qquad \boldsymbol{A}_2=\begin{pmatrix} 0 & A_{16} & A_{17} & A_{18} \\ -\frac{\sqrt{105}}{7}A_{16} & -\frac{4\sqrt{21}}{21}A_{17} & A_{27} & A_{28} \\ \frac{\sqrt{105}}{21}A_{17} & -\frac{5}{3}A_{27}-\frac{2}{3}A_{18} & -\frac{4\sqrt{21}}{21}A_{28} & A_{38} \\ A_{18}+2A_{27} & \frac{\sqrt{105}}{21}A_{28} & -\frac{\sqrt{105}}{7}A_{38} & 0 \\ 0 & \frac{2\sqrt{35}}{7}A_{38} & 0 & 0 \\ -\frac{2\sqrt{35}}{7}A_{38} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

3)当 $\boldsymbol{\Phi}=\boldsymbol{\Phi}_2^{(7)}$ 或 $\boldsymbol{\Phi}=\boldsymbol{\Phi}_5^{(7)}$,此时 $\boldsymbol{A}$ 表达式如下

$$
\boldsymbol{A}=\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{18}\\
0 & 0 & 0 & 0 & 0 & 0 & -A_{18} & 0\\
0 & 0 & 0 & 0 & 0 & A_{18} & 0 & 0\\
0 & 0 & 0 & 0 & -A_{18} & 0 & 0 & 0\\
0 & 0 & 0 & A_{18} & 0 & 0 & 0 & 0\\
0 & 0 & -A_{18} & 0 & 0 & 0 & 0 & 0\\
0 & A_{18} & 0 & 0 & 0 & 0 & 0 & 0\\
-A_{18} & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
$$

4) 当 $\boldsymbol{\Phi}=\boldsymbol{\Phi}_3^{(7)}$ 或 $\boldsymbol{\Phi}=\boldsymbol{\Phi}_4^{(7)}$，此时 $\boldsymbol{A}$ 表达式为

$$\boldsymbol{A}=0.$$

接下来，将通过给出上述的解来构造$\mathbb{H}P^n$ 中极小二维球面。

式(1)对应的解

设 $\boldsymbol{U}=\begin{pmatrix}\boldsymbol{U}_1 & 0\\ 0 & \boldsymbol{I}_4\end{pmatrix}$，那么等式(1) 等价于 $\begin{pmatrix}0 & \boldsymbol{U}_1^{\mathrm{T}}\\ -\boldsymbol{U}_1 & 0\end{pmatrix}=\boldsymbol{A}$，令 $A_{14}=A_{15}=A_{16}=A_{25}=A_{38}=A_{47}=A_{48}=A_{58}=0$，

则 $\boldsymbol{U}_1=\begin{pmatrix}
0 & 0 & -\dfrac{\sqrt{105}}{35}A_{17}-\dfrac{2\sqrt{5}}{5}A_{26} & -\dfrac{1}{5}A_{18}-A_{27}-\dfrac{9}{5}A_{36}\\
0 & A_{26} & A_{36} & -\dfrac{\sqrt{105}}{35}A_{28}-\dfrac{2\sqrt{5}}{5}A_{37}\\
A_{17} & A_{27} & A_{37} & 0\\
A_{18} & A_{28} & 0 & 0
\end{pmatrix}$.

取 $A_{26}=A_{37}=A_{17}=A_{28}=0$，$A_{18}=A_{27}=A_{36}=1$，所以

$$
\boldsymbol{U}=\begin{pmatrix}
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}\in \boldsymbol{U}(8)
$$

为式(1)对应的特解。

此时可算得

$$
\pi\circ(\boldsymbol{U}\cdot\boldsymbol{\Psi}_0^{(7)})=\left[\begin{pmatrix}-\sqrt{35}z^3\\ \sqrt{21}z^2\\ -\sqrt{7}z\\ 1\end{pmatrix}+\mathrm{j}\begin{pmatrix}\sqrt{35}z^4\\ \sqrt{21}z^5\\ \sqrt{7}z^6\\ z^7\end{pmatrix}\right]_{\mathbb{H}},\quad \pi\circ(\boldsymbol{U}\cdot\boldsymbol{\Psi}_7^{(7)})=\left[\begin{pmatrix}-\sqrt{35}\bar{z}^4\\ -\sqrt{21}\bar{z}^5\\ -\sqrt{7}\bar{z}^6\\ -\bar{z}^7\end{pmatrix}+\mathrm{j}\begin{pmatrix}-\sqrt{35}\bar{z}^3\\ \sqrt{21}\bar{z}^2\\ -\sqrt{7}\bar{z}\\ 1\end{pmatrix}\right]_{\mathbb{H}}.
$$

式(2)对应的解

设 $\boldsymbol{U}=\begin{pmatrix}\boldsymbol{U}_1 & 0\\ 0 & \boldsymbol{I}_4\end{pmatrix}$，等式(2) 等价于$\begin{pmatrix}0 & \boldsymbol{U}_1^{\mathrm{T}}\\ -\boldsymbol{U}_1 & 0\end{pmatrix}=\boldsymbol{A}$，令 $A_{16}=A_{38}=0$，则

$$\boldsymbol{U}_1=\begin{pmatrix} 0 & 0 & \frac{\sqrt{105}}{21}A_{17} & A_{18}+2A_{27} \\ 0 & -\frac{4\sqrt{21}}{21}A_{17} & -\frac{5}{3}A_{27}-\frac{2}{3}A_{18} & \frac{\sqrt{105}}{21}A_{28} \\ A_{17} & A_{27} & -\frac{4\sqrt{21}}{21}A_{28} & 0 \\ A_{18} & A_{28} & 0 & 0 \end{pmatrix}$$

.取 $A_{17}=A_{28}=0, A_{18}=1, A_{27}=-1$,所以

$$\boldsymbol{U}=\begin{pmatrix} 0&0&0&-1&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&-1&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1 \end{pmatrix}\in \boldsymbol{U}(8)$$

为式(2)对应的特解。

此时可算得

$$\pi\circ(U\cdot\boldsymbol{\varPsi}_1^{(7)})=\left[\begin{pmatrix} -\sqrt{35}(3z^2-4z^3\bar{z}) \\ \sqrt{21}(2z-5z^2\bar{z}) \\ -\sqrt{7}(1-6z\bar{z}) \\ -7\bar{z} \end{pmatrix}+\mathrm{j}\begin{pmatrix} \sqrt{35}(4z^3-3z^4\bar{z}) \\ \sqrt{21}(5z^4-2z^5\bar{z}) \\ \sqrt{7}(6z^5-z^6\bar{z}) \\ 7z^6 \end{pmatrix}\right]_{\mathrm{H}},$$

$$\pi\circ(\boldsymbol{U}\cdot\boldsymbol{\varPsi}_6^{(7)})=\left[\begin{pmatrix} -\sqrt{35}(-4\bar{z}^3+3z\bar{z}^4) \\ \sqrt{21}(5\bar{z}^4-2z\bar{z}^5) \\ -\sqrt{7}(-6\bar{z}^5+z\bar{z}^6) \\ 7\bar{z}^6 \end{pmatrix}+\mathrm{j}\begin{pmatrix} \sqrt{35}(3\bar{z}^2-4z\bar{z}^3) \\ \sqrt{21}(-2\bar{z}+5z\bar{z}^2) \\ \sqrt{7}(1-6z\bar{z}) \\ 7z \end{pmatrix}\right]_{\mathrm{H}}.$$

式(3)对应的解

设 $\boldsymbol{U}=\begin{pmatrix} \boldsymbol{U}_1 & 0 \\ 0 & \boldsymbol{I}_4 \end{pmatrix}$,等式(2) 等价于 $\begin{pmatrix} 0 & \boldsymbol{U}_1^{\mathrm{T}} \\ -\boldsymbol{U}_1 & 0 \end{pmatrix}=\boldsymbol{A}$,令 $\boldsymbol{A}_{18}=1$,则

$$\boldsymbol{U}=\begin{pmatrix} 0&0&0&-1&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&-1&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1 \end{pmatrix}\in \boldsymbol{U}(8)$$

为式(3)对应的特解。

此时可算得

$$\pi\circ(\boldsymbol{U}\cdot\boldsymbol{\varPsi}_2^{(7)})=\left[\begin{pmatrix} -\sqrt{35}(3z-12z^2\bar{z}+6z^3\bar{z}^2) \\ \sqrt{21}(1-10z\bar{z}+10z^2\bar{z}^2) \\ -\sqrt{7}(-6\bar{z}+15z\bar{z}^2) \\ 21\bar{z}^2 \end{pmatrix}+\mathrm{j}\begin{pmatrix} \sqrt{35}(6z^2-12z^3\bar{z}+3z^4\bar{z}^2) \\ \sqrt{21}(10z^3-10z^4\bar{z}+z^5\bar{z}^2) \\ \sqrt{7}(15z^4-6z^5\bar{z}) \\ 21z^5 \end{pmatrix}\right]_{\mathrm{H}},$$

$$\pi\circ(\boldsymbol{U}\cdot\boldsymbol{\Psi}_5^{(7)})=\left[\begin{pmatrix}-\sqrt{35}(6\bar z^2-12z\bar z^3+3z^2\bar z^4)\\ \sqrt{21}(-10\bar z^3+10z\bar z^4-z^2\bar z^5)\\ -\sqrt{7}(15\bar z^4-6z\bar z^5)\\ -21\bar z^5\end{pmatrix}+\mathrm{j}\begin{pmatrix}\sqrt{35}(-3\bar z+12z\bar z^2-6z^2\bar z^3)\\ \sqrt{21}(1-10z\bar z+10z^2\bar z^2)\\ \sqrt{7}(6z-15z^2\bar z)\\ 21z^2\end{pmatrix}\right]_{\mathbb{H}}.$$

式(4)对应的解

由于 $\boldsymbol{A}=0$ 非退化,所以此方程无解。

例 2 设 z 为球面上一个全纯坐标,则下述为 6 个 $\mathbb{H}P^3$ 中常曲率共形极小二维球面:

1) $\pi\circ(\boldsymbol{U}\cdot\boldsymbol{\Psi}_0^{(7)})=\left[\begin{pmatrix}-\sqrt{35}z^3\\ \sqrt{21}z^2\\ -\sqrt{7}z\\ 1\end{pmatrix}+\mathrm{j}\begin{pmatrix}\sqrt{35}z^4\\ \sqrt{21}z^5\\ \sqrt{7}z^6\\ z^7\end{pmatrix}\right]_{\mathbb{H}}$ 和 $\pi\circ(U\cdot\boldsymbol{\Psi}_7^{(7)})=\left[\begin{pmatrix}-\sqrt{35}\bar z^4\\ -\sqrt{21}\bar z^5\\ -\sqrt{7}\bar z^6\\ -\bar z^7\end{pmatrix}+\mathrm{j}\begin{pmatrix}-\sqrt{35}\bar z^3\\ \sqrt{21}\bar z^2\\ -\sqrt{7}\bar z\\ 1\end{pmatrix}\right]_{\mathbb{H}}$ 为曲率是 4/7 的共形极小二维球面。

2) $\pi\circ(\boldsymbol{U}\cdot\boldsymbol{\Psi}_1^{(7)})=\left[\begin{pmatrix}-\sqrt{35}(3z^2-4z^3\bar z)\\ \sqrt{21}(2z-5z^2\bar z)\\ -\sqrt{7}(1-6z\bar z)\\ -7\bar z\end{pmatrix}+\mathrm{j}\begin{pmatrix}\sqrt{35}(4z^3-3z^4\bar z)\\ \sqrt{21}(5z^4-2z^5\bar z)\\ \sqrt{7}(6z^5-z^6\bar z)\\ 7z^6\end{pmatrix}\right]_{\mathbb{H}}$ 和 $\pi\circ(\boldsymbol{U}\cdot\boldsymbol{\Psi}_6^{(7)})=\left[\begin{pmatrix}-\sqrt{35}(-4\bar z^3+3z\bar z^4)\\ \sqrt{21}(5\bar z^4-2z\bar z^5)\\ -\sqrt{7}(-6\bar z^5+z\bar z^6)\\ 7\bar z^6\end{pmatrix}+\mathrm{j}\begin{pmatrix}\sqrt{35}(3\bar z^2-4z\bar z^3)\\ \sqrt{21}(-2\bar z+5z\bar z^2)\\ \sqrt{7}(1-6z\bar z)\\ 7z\end{pmatrix}\right]_{\mathbb{H}}$ 为曲率是 4/19 的共形极小二维球面。

3) $\pi\circ(\boldsymbol{U}\cdot\boldsymbol{\Psi}_2^{(7)})=\left[\begin{pmatrix}-\sqrt{35}(3z-12z^2\bar z+6z^3\bar z^2)\\ \sqrt{21}(1-10z\bar z+10z^2\bar z^2)\\ -\sqrt{7}(-6\bar z+15z\bar z^2)\\ 21\bar z^2\end{pmatrix}+\mathrm{j}\begin{pmatrix}\sqrt{35}(6z^2-12z^3\bar z+3z^4\bar z^2)\\ \sqrt{21}(10z^3-10z^4\bar z+z^5\bar z^2)\\ \sqrt{7}(15z^4-6z^5\bar z)\\ 21z^5\end{pmatrix}\right]_{\mathbb{H}}$ 和 $\pi\circ(\boldsymbol{U}\cdot\boldsymbol{\Psi}_5^{(7)})=\left[\begin{pmatrix}-\sqrt{35}(6\bar z^2-12z\bar z^3+3z^2\bar z^4)\\ \sqrt{21}(-10\bar z^3+10z\bar z^4-z^2\bar z^5)\\ -\sqrt{7}(15\bar z^4-6z\bar z^5)\\ -21\bar z^5\end{pmatrix}+\mathrm{j}\begin{pmatrix}\sqrt{35}(-3\bar z+12z\bar z^2-6z^2\bar z^3)\\ \sqrt{21}(1-10z\bar z+10z^2\bar z^2)\\ \sqrt{7}(6z-15z^2\bar z)\\ 21z^2\end{pmatrix}\right]_{\mathbb{H}}$ 为曲率是 4/27 的共形极小二维球面。

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