

Convolution integral restricted on closed hypersurfaces^{*}

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Abstract The classical convolution integral on Euclidean space is given as follows. For $f \in L^1(\mathbb{R}^n)$ and $g \in L^p(\mathbb{R}^n)$, $T_f(g)$ is defined as

$$T_f(g)(x) := f * g(x) = \int_{\mathbb{R}^n} f(x-y)g(y)dy.$$

It has many applications in analysis and engineering. Young's inequality demonstrates that $T_f: L^p(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n)$ is a bounded operator for $1 \leq p \leq \infty$. In this study, we have obtained the estimation of the L^p norm of convolution integral restricted on closed hypersurfaces. More precisely, we have established Young's inequality on closed hypersurfaces.

Keywords convolution integral; closed hypersurface; boundedness

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限制在闭超曲面上的卷积

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摘 要 经典的欧氏空间中的卷积如下给出。对 $f \in L^1(\mathbb{R}^n)$ 和 $g \in L^p(\mathbb{R}^n)$,

$$T_f(g)(x) := f * g(x) = \int_{\mathbb{R}^n} f(x-y)g(y)dy.$$

这样的卷积在分析、物理和工程上都有广泛的应用。经典的 Young 不等式表明, 对 $1 \leq p \leq \infty$, $T_f: L^p(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n)$ 是有界线性算子。得到限制在一个闭超曲面(欧氏空间中的余维数为 1 的紧致无边连通正则子流形)上的卷积的 L^p 模估计的大小。更精确地说, 把 Young 不等式推广到了闭超曲面上。

关键词 卷积; 闭超曲面; 有界性

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The classical concept of convolution operator has been generalized in many new cases. The reason is that convolution operator has many applications in harmonic analysis and engineering. For example, it can be used to characterize the bounded operators which commute with transition actions.

Many researchers have made explorations in these topics. For instance, Oinarov^[1] explored the boundedness and compactness of convolution operators of fractional integration type. Avsyankin^[2] and Guliyeva and Sadigova^[3] explored the properties of convolution operators on Morrey spaces.

Harmonic analysis on Euclidean space has developed very fast. It is also meaningful to generalize the theories on manifolds. For example, the progress of restriction conjecture about Fourier transformation has been introduced in Refs. [4–7]. Similarly, we consider the restriction properties of convolution integral on manifolds in this study.

1 Some definitions

Before we put forward our main results, some useful definitions are given as follows.

Definition 1.1 Sobolev space $W_1^k(\mathbb{R}^n)$ is defined as

$$W_1^k(\mathbb{R}^n) = \{f \in L^1(\mathbb{R}^n) : \partial^\alpha f \in L^1(\mathbb{R}^n); |\alpha| \leq k\}.$$

For $f \in W_1^k(\mathbb{R}^n)$, the norm of f is defined as

$$\|f\|_{W_1^k} = \|f\|_1 + \sum_{|\alpha| \leq k} \|\partial^\alpha f\|_1.$$

Definition 1.2 Suppose M is an $(n-1)$ dimensional hypersurface in \mathbb{R}^n . For $f \in W_1^1(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ and $g \in L^p(\mathbb{R}^n)$, we define

$$\|T_f(g)\|_{p,M}^p := \int_M |T_f(g)(x)|^p d\sigma(x), \quad (1)$$

where, $p \geq 1$ and $d\sigma$ is the surface measure of M in \mathbb{R}^n .

2 Main result

We state our main theorem as follows.

Theorem 2.1 Let M be a closed (connected compact without boundary) $(n-1)$ dimensional hypersurface in \mathbb{R}^n . Then, for $1 \leq p \leq \infty$, $f \in W_1^1(\mathbb{R}^n) \cap C(\mathbb{R}^n)$, and $g \in L^p(\mathbb{R}^n)$, the inequality

$$\|T_f(g)\|_{p,M} \leq C'(M) \|f\|_{W_1^1} \|g\|_p$$

holds. Here, $C'(M)$ is a constant relying on M .

3 Proof of the main result

According to Lemma 3.1 (see below), we only need to prove Theorem 2.1 for all $f \in W_1^1(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$ and $g \in L^p(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$.

Lemma 3.1 Let M be a closed $(n-1)$ dimensional hypersurface in \mathbb{R}^n . If the following inequality holds for all $f \in W_1^1(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$ and $g \in L^p(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$,

$$\|T_f(g)\|_{p,M} \leq C(M) \|f\|_{W_1^1} \|g\|_p \quad (2)$$

also holds for all $f \in W_1^1(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ and $g \in L^p(\mathbb{R}^n)$.

Proof:

In fact, under the hypothesis in the lemma, given any $f \in W_1^1(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ and $g \in L^p(\mathbb{R}^n)$, we can assume $f_k \in W_1^1(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$ and $g_k \in L^p(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$ such that

$$\|f_k - f\|_{W_1^1} \rightarrow 0, \quad (3)$$

and

$$\|g_k - g\|_p \rightarrow 0. \quad (4)$$

Without loss of generality, we are able to assume that the sequence f_k converges to f almost everywhere and g_k converges to g almost everywhere.

Thus, applying Fatou's lemma and using (2), (3), and (4), we have

$$\begin{aligned} \|T_f(g)\|_{p,M} &= \|f * g\|_{p,M} \\ &\leq \| |f| * |g| \|_{p,M} \\ &= \left\| \int_{\mathbb{R}^n} \lim_{k \rightarrow \infty} |f_k|(x-y) |g_k|(y) dy \right\|_{p,M} \\ &\leq \liminf_{k \rightarrow \infty} \| |f_k| * |g_k| \|_{p,M} \\ &\leq \liminf_{k \rightarrow \infty} C(M) \|f_k\|_{W_1^1} \|g_k\|_p \\ &= C(M) \|f\|_{W_1^1} \|g\|_p. \end{aligned}$$

This completes the proof of Lemma 3.1.

Then, we state the following tubular neighborhood lemma^[8].

Lemma 3.2 Let S be a closed hypersurface in Euclidean space. (N, S, π, \mathbb{R}) is the normal bundle of S . Then, there exists a $\delta > 0$ and a tubular neighborhood $\Delta_\delta = \{(p, \eta) \in N : \|\eta\| < \delta\}$, such that Δ_δ is diffeomorphic to $N_\delta = \{x \in N_p \subset \mathbb{R}^{n+1} : p \in S; d$

$(x, S) < \delta$ under the mapping $\phi(p, \eta) = p + \eta$. Thus, for each two points p and q on S , the corresponding normal lines passing through these two points and having these two points as the lines' centers do not intersect, and these normal lines have length of 2δ .

Then, because M is a closed hypersurface in \mathbb{R}^n , according to generalized Jordan separation theorem^[8] we can assume that D is a bounded open domain in \mathbb{R}^n , whose boundary is M , i.e., $\partial D = M$. Now, we have the following lemma.

Lemma 3.3 Let M be a $(n - 1)$ dimensional closed hypersurface in \mathbb{R}^n . Then, for $1 \leq p \leq \infty$, the inequality

$$\|T_f(g)\|_{p,M} \leq C(M) \|f\|_{W_1^1} \|g\|_p$$

holds for all $f \in W_1^1(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$ and $g \in L^p(\mathbb{R}^n)$.

Proof of Lemma 3.3 and Theorem 2.1:

We first prove that the inequality holds for $1 < p < \infty$. Since $f(x - y)g(y) = (f_1(x - y) + if_2(x - y))(g_1(y) + ig_2(y))$, we are able to suppose f and g are real valued functions.

Let Ω be the volume form on M , and let $i: M \hookrightarrow \mathbb{R}^n$ be the standard embedding. Δ_δ is the tubular neighborhood of M . According to Lemma 3.2, there exists a smooth unit vector field \mathbf{n} on Δ_δ . We fix a smooth function ϕ valuing 1 in the closure $\Delta_{\frac{\delta}{2}}$ and valuing 0 outside Δ_δ . Let $\eta = \phi \mathbf{n}$ be the smooth vector field on \mathbb{R}^n . Then, the volume form Ω in Ref. [9] can be represented as

$$\Omega = i^* \sum_{\alpha=1}^n (-1)^{\alpha-1} \eta^\alpha(x) dx^1 \wedge \cdots \wedge \widehat{dx^\alpha} \wedge \cdots \wedge dx^n, \quad (5)$$

where η^α is the α th component of η .

Therefore, substituting surface measure $d\sigma$ in (1) by volume form Ω in (5), we obtain

$$\begin{aligned} \|T_f(g)\|_{p,M}^p &= \int_M |T_f(g)(x)|^p d\sigma(x) \\ &= \int_M |T_f(g)(x)|^p \Omega \\ &= \sum_{\alpha=1}^n \int_M i^* h_0 (-1)^{\alpha-1} \eta^\alpha dx^1 \wedge \cdots \wedge \widehat{dx^\alpha} \wedge \cdots \wedge dx^n, \end{aligned} \quad (6)$$

where $h_0(x) = |T_f(g)(x)|^p$.

Since, for $p > 1$, $h_0(x)$ is smooth for $\varepsilon > 0$ by

Sard theorem^[9], there exists a $c \in \mathbb{R}$ such that $|c| < \varepsilon$ and 0 is the regular value of $T_f(g)(x) - c$. This means that the gradient of $T_f(g)(x) - c$ at the zeros of this function does not vanish. Let $h(x)$ be $|T_f(g)(x) - c|^p$, we have

$$\begin{aligned} \|T_f(g)\|_{p,M}^p &= \int_M |T_f(g)(x) - c + c|^p d\sigma(x) \\ &\leq 2^{p-1} \varepsilon^p |M| + 2^{p-1} \int_M h(x) d\sigma(x). \end{aligned} \quad (7)$$

Here, we have used inequality (8).

$$(a + b)^p \leq 2^{p-1} (a^p + b^p). \quad (8)$$

According to the regular value preimage theorem^[10], $\Gamma = \{h(x) = 0\}$ is a $(n-1)$ dimensional regular submanifold in \mathbb{R}^n , whose Lebesgue measure is zero. Take Γ_ε be the ε tubular neighborhood of Γ such that

$$\int_{\Gamma_\varepsilon} \left| \frac{\partial h}{\partial x^\alpha} \right| dx < \varepsilon. (\alpha \in \{1, \dots, n\}). \quad (9)$$

Then, we use (6) and apply Stokes formula.

Because $\eta^\alpha(x)$ and $\frac{\partial \eta^\alpha(x)}{\partial x^\alpha}$ are bounded over D , we obtain

$$\begin{aligned} & \left| \int_M h(x) d\sigma(x) \right| \\ &= \left| \int_M i^* h (-1)^{\alpha-1} \eta^\alpha dx^1 \wedge \cdots \wedge \widehat{dx^\alpha} \wedge \cdots \wedge dx^n \right| \\ &= \left| \int_D \frac{\partial h}{\partial x^\alpha} \eta^\alpha dx + \int_D h \frac{\partial \eta^\alpha}{\partial x^\alpha} dx \right| \\ &\leq C(M) \left(\int_D \left| \frac{\partial h}{\partial x^\alpha} \right| dx + \int_D |h| dx \right), \end{aligned} \quad (10)$$

where $C(M) = \max \left\{ \eta^\alpha(x), \frac{\partial \eta^\alpha(x)}{\partial x^\alpha} : x \in D, \alpha = 1, \dots, n \right\}$.

Next, we estimate the two parts in (10) separately. Using (9), we first have

$$\begin{aligned} \int_D \left| \frac{\partial h}{\partial x^\alpha} \right| dx &= \left(\int_{D-\Gamma_\varepsilon} + \int_{\Gamma_\varepsilon} \right) \left| \frac{\partial h}{\partial x^\alpha} \right| dx \\ &\leq \varepsilon + \int_{D-\Gamma_\varepsilon} \left| \frac{\partial h}{\partial x^\alpha} \right| dx \\ &= \varepsilon + \int_{D-\Gamma_\varepsilon} \left| \frac{\partial}{\partial x^\alpha} |T_f(g) - c|^p \right| dx \\ &= \int_{D-\Gamma_\varepsilon} p |T_f(g) - c|^{p-1} \left| \frac{\partial |T_f(g) - c|}{\partial x^\alpha} \right| dx + \varepsilon. \end{aligned} \quad (11)$$

Since $|T_{\frac{\partial f}{\partial x^\alpha}}(g)| = |\frac{\partial}{\partial x^\alpha} T_f(g) - c|$ on $D - \Gamma_\varepsilon$, applying Hölder inequality and (8) in (11), we obtain

$$\begin{aligned} \int_D |\frac{\partial h}{\partial x^\alpha}| dx &\leq \varepsilon + \int_{D-\Gamma_\varepsilon} p |T_f(g) - c|^{p-1} |T_{\frac{\partial f}{\partial x^\alpha}}(g)| dx \\ &\leq \varepsilon + \int_D p |T_f(g) - c|^{p-1} |T_{\frac{\partial f}{\partial x^\alpha}}(g)| dx \\ &\leq \varepsilon + p \|T_f(g) - c\|_{p,D}^{p-1} \|T_{\frac{\partial f}{\partial x^\alpha}}(g)\|_{p,D} \\ &\leq \varepsilon + 2^{(p-1)^2} p |D|^{\frac{p-1}{p}} \varepsilon^{p-1} \|T_{\frac{\partial f}{\partial x^\alpha}}(g)\|_p + \\ &\quad 2^{(p-1)^2} p \|T_f(g)\|_p^{p-1} \|T_{\frac{\partial f}{\partial x^\alpha}}(g)\|_p. \end{aligned} \quad (12)$$

Meanwhile, applying the following Young's inequalities

$$\begin{aligned} \|T_f(g)\|_p &\leq \|f\|_1 \|g\|_p, \\ \|T_{\frac{\partial f}{\partial x^\alpha}}(g)\|_p &\leq \|\frac{\partial f}{\partial x^\alpha}\|_1 \|g\|_p, \end{aligned}$$

we have

$$\begin{aligned} \int_D |h| dx &\leq 2^{p-1} (\int_D |T_f(g)(x)|^p dx + \varepsilon^p |D|) \\ &\leq 2^{p-1} \varepsilon^p |D| + 2^{p-1} \|f\|_1^p \|g\|_p^p, \end{aligned} \quad (13)$$

and

$$\begin{aligned} \int_D \left| \frac{\partial h}{\partial x^\alpha} \right| dx &\leq \varepsilon + 2^{(p-1)^2} p |D|^{\frac{p-1}{p}} \varepsilon^{p-1} \|T_{\frac{\partial f}{\partial x^\alpha}}(g)\|_p + \\ &\quad 2^{(p-1)^2} p \|g\|_p^p \|f\|_1^{p-1} \|\frac{\partial f}{\partial x^\alpha}\|_1. \end{aligned} \quad (14)$$

Combining (7), (10), (13), and (14), we have

$$\begin{aligned} \|T_f(g)\|_{p,M}^p &\leq 2^{p-1} \varepsilon^p |M| + 2^{p-1} \int_M h(x) d\sigma(x) \\ &\leq 2^{p-1} \varepsilon^p |M| + 2^{p-1} C(M) (2^{p-1} \varepsilon^p |D| + \varepsilon + \\ &\quad (2^{p-1} + 2^{(p-1)^2} p) \|f\|_1^{p-1} \|f\|_{w_1^1} \|g\|_p^p) \\ &\leq 2^{p-1} \varepsilon^p |M| + 2^{p-1} C(M) (2^{p-1} \varepsilon^p |D| + \varepsilon + \\ &\quad 2^{(p-1)^2} p |D|^{\frac{p-1}{p}} \varepsilon^{p-1} \|T_{\frac{\partial f}{\partial x^\alpha}}(g)\|_p) + \\ &\quad 2^{p-1} (2^{p-1} + 2^{(p-1)^2} p) C(M) \|f\|_{w_1^1}^p \|g\|_p^p. \end{aligned} \quad (15)$$

For arbitrary ε and $p > 1$, let $\varepsilon \rightarrow 0$ in (15).

We obtain

$$\|T_f(g)\|_{p,M}^p \leq C(p, M) \|f\|_{w_1^1}^p \|g\|_p^p. \quad (16)$$

Now, we have finished the proof in the case

where $1 < p < \infty$.

For $p = 1$, we choose $g \in C_c^\infty(\mathbb{R}^n)$. Let $p \rightarrow 1$ in (16), by Fatou's lemma, we obtain

$$\|T_f(g)\|_{1,M} \leq 2C(M) \|f\|_{w_1^1} \|g\|_1. \quad (17)$$

Because $C_c^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$, by Fatou's lemma, we finish the proof for $p = 1$.

Meanwhile, it is obvious that

$$\|T_f(g)\|_{\infty,M} \leq \|f\|_1 \|g\|_\infty \quad (18)$$

holds.

Finally, using (17), (18), and Riesz-Thörin interpolation theorem^[11], we can choose a constant $C'(M) = \max\{2C(M), 1\}$ which is independent of p such that the following is true.

$$\|T_f(g)\|_{p,M} \leq C'(M) \|f\|_{w_1^1} \|g\|_p.$$

This completes the proof of Lemma 3.3. Due to Lemma 3.1, we finish the proof of Theorem 2.1.

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