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Sharp bounds for fractional Hardy operator on higher-dimensional product spaces*

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Abstract In this paper, we get the sharp bounds for fractional Hardy operator on higher-dimensional product spaces from $L^1(\mathbb{R}^{n_1}\times\cdots\times\mathbb{R}^{n_m})$ to the space $wL^Q(\mathbb{R}^{n_1}\times\cdots\times\mathbb{R}^{n_m})$. More generally, the norm of fractional Hardy operator on higher-dimensional product spaces from $L^P(\mathbb{R}^{n_1}\times\cdots\times\mathbb{R}^{n_m})$ to $L^{Q_I}(\mathbb{R}^{n_1}\times\cdots\times\mathbb{R}^{n_m})$ is obtained.

Keywords fractional Hardy operator; operator norm; product space; L^{Q_I}

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高维乘积空间上分数次 Hardy 算子的最佳界

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摘 要 得到高维乘积空间上分数次 Hardy 算子从 $L^1(\mathbb{R}^{n_1}\times\cdots\times\mathbb{R}^{n_m})$ 到 $wL^Q(\mathbb{R}^{n_1}\times\cdots\times\mathbb{R}^{n_m})$ 的最佳界。更一般地,还得到高维乘积空间上分数次 Hardy 算子从 $L^P(\mathbb{R}^{n_1}\times\cdots\times\mathbb{R}^{n_m})$ 到 $L^{Q_I}(\mathbb{R}^{n_1}\times\cdots\times\mathbb{R}^{n_m})$ 的算子范数。

关键词 分数次哈代算子;算子范数;乘积空间; L^{Q_I}

Let f be a non-negative integrable function on $\mathbb{R}^n$. The fractional Hardy operator on $\mathbb{R}^n$ is defined by

$$\mathbb{H}_\beta f(x) := \frac{1}{|B(0,|x|)|^{1-\frac{\beta}{n}}}\int_{|y|<|x|} f(y)\,\mathrm{d}y, \tag{1}$$

for $x\in\mathbb{R}^n$.

For weak and strong operator norms of fractional Hardy operator, the following Theorem due to Lu, Zhao, and Yan[1] is well-known.

Theorem A Suppose $0<\beta<n$, $1<p<q<\frac{n}{\beta}$, and $\frac{1}{p}-\frac{1}{q}=\frac{\beta}{n}$.

(i) If $f\in L^p(\mathbb{R}^n)$, we have

$$\|\mathbb{H}_\beta f\|_{L^q(\mathbb{R}^n)} \leqslant C\|f\|_{L^p(\mathbb{R}^n)},$$

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where

$$\left(\frac{p}{q}\right)^{\frac{1}{q}}\left(\frac{p}{p-1}\right)^{\frac{1}{q}}\left(\frac{q}{q-1}\right)^{1-\frac{1}{q}}\left(1-\frac{p}{q}\right)^{\frac{1}{p}-\frac{1}{q}} \leqslant C \leqslant \left(\frac{p}{p-1}\right)^{\frac{p}{q}}.$$

(ii) If $f \in L^1(\mathbb{R}^n)$, then for any $\lambda > 0$,

$$|\{x \in \mathbb{R}^n : |\mathbb{H}_\beta(f)(x)| > \lambda\}| \leqslant \left(\frac{1}{\lambda}\|f\|_{L^1}\right)^{\frac{n}{n-\beta}}.$$

Moreover,

$$\|\mathbb{H}_\beta\|_{L^1(\mathbb{R}^n)\to L^{\frac{n}{n-\beta},\infty}(\mathbb{R}^n)} = 1.$$

Based on the previous work[1], Zhao and Lu[2] obtained the operator norm of fractional Hardy operators on $\mathbb{R}^n$.

Theorem B Suppose that $0 < \beta < n$, $1 < p < q < \infty$, and $\frac{1}{p} - \frac{1}{q} = \frac{\beta}{n}$. If $f \in L^p(\mathbb{R}^n)$, we have

$$\|\mathbb{H}_\beta f\|_{L^q(\mathbb{R}^n)} \leqslant A\|f\|_{L^p(\mathbb{R}^n)}.$$

Moreover,

$$\|\mathbb{H}_\beta\|_{L^q(\mathbb{R}^n)\to L^q(\mathbb{R}^n)} = A,$$

where

$$A = \left(\frac{p'}{q}\right)\left(\frac{n}{q\beta}\cdot B\left(\frac{n}{q\beta},\frac{n}{q'\beta}\right)\right)^{\frac{-\beta}{n}}.$$

In recent years, Hardy operator on product space has received much concern. In 2012, Wang, et al[3] introduced Hardy operator on product space and obtained the operator norm. Subsequently, Lu, et al[1] generalized Wang's work to higher-dimensional product space. Very recently, He, et al[4] investigated Hardy type operators on higher-dimensional product spaces and obtained the sharp bounds. Inspired by Refs. [1-4], we will compute the operator norm of fractional Hardy operator on higher-dimensional product space in the present paper.

Now we are in a position to introduce fractional Hardy operator on higher-dimensional product space. Let f be a non-negative integrable function on $\mathbb{R}^{n_1}\times\cdots\times\mathbb{R}^{n_m}$ and then the fractional Hardy operator on product space is defined by

$$\mathbb{H}_{\beta_1,\cdots,\beta_m} f(x) := \frac{1}{|B(0,|x_1|)|^{1-\frac{\beta_1}{n_1}}}\cdots\frac{1}{|B(0,|x_m|)|^{1-\frac{\beta_m}{n_m}}}\times\int_{|y_1|<|x_1|}\cdots\int_{|y_m|<|x_m|} f(y_1,\cdots,y_m)\,dy_1\cdots dy_m \tag{2}$$

for $x = (x_1, x_2, \cdots, x_m) \in \mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\times\cdots\times\mathbb{R}^{n_m}$ and $0 < \beta_i < n_i$, $i = 1,\cdots,m$.

1 Preliminaries

Before we present the main results, some useful lemmas and definitions are needed.

The mixed norm space was first defined in Ref. [5] by Benedek and Panzone and received much concern (See Refs. [6-10]). In 2018, Wei and Yan[11] defined a more general mixed norm space which is called weak and strong mixed-norm space. We list its definition for completeness.

Definition 1.1 Let (X_i, S_i, μ_i), for $1 \leqslant i \leqslant n$, be n given, totally σ-finite measure spaces and $\boldsymbol{P} = (p_1, p_2, \cdots, p_n)$ a given n-tuple with $1 \leqslant p_i \leqslant \infty$. The set I satisfies $I \subset \{1,\cdots,n\}$. A function $f(x_1, x_2, \cdots, x_n)$ measurable in the product spaces $(X, S, \mu) = \left(\prod_{i=1}^n X_i, \prod_{i=1}^n S_i, \prod_{i=1}^n \mu_i\right)$ is said to belong to the space $L^{\boldsymbol{P}_I}(X)$ if the number obtained after subsequently taking successfully the mixed norm where we take p_i − norm for $i \in I$ while we take weak p_j − norm for $j \in \{1,\cdots,n\}\backslash I$ and in natural order is finite. The number so obtained, finite or not, will be denoted by $\|f\|_{\boldsymbol{P}_I}$.

We give some necessary remarks for the space $L^{\boldsymbol{P}_I}(X)$. For more properties, we refer readers to Ref. [11].

(i) If $I = \{1,\cdots,n\}$, we call $L^{\boldsymbol{P}_I}(X)$ strong mixed norm space, which is also denoted by $L^{\boldsymbol{P}}(X)$ or $L^{p_1,\cdots,p_n}(X)$.

(ii) If the set I is empty, we call $L^{\boldsymbol{P}_I}(X)$ weak mixed norm space, which is also denoted by $wL^{\boldsymbol{P}}(X)$ or $wL^{p_1,\cdots,p_n}(X)$.

(iii) The space $L^{\boldsymbol{P}_I}(X)$ is a quasi-normed space for $P \geqslant \boldsymbol{1}$.

For the mixed norm, we have a basic lemma which plays an important role in the proof of our main theorems.

Lemma 1.1 Let (X, S, μ) be defined as in the above definitions. If $p_n \geqslant \cdots \geqslant p_1 \geqslant 1$ and $f \in L^{p_n,\cdots,p_1}(X)$, then $f \in L^{p_1,\cdots,p_n}(X)$ and there holds

$$\|f\|_{L^{p_1,\cdots,p_n}(X)} \leqslant \|f\|_{L^{p_n,\cdots,p_1}(X)}.$$

Lemma 1.1 is a direct generation of the Minkowski's inequality. For the proof of the lemma, readers are referred to Ref. [12]. It is not hard to see Fubini's theorem is a special case of Lemma 1.1.

In the rest part of our paper, we always consider the spaces on Euclidean space.

2 Main results and proof

Now we formulate our main theorems. We first give the boundedness of Hardy operator on product space from $L^1(\mathbb{R}^{n_1}\times\cdots\times\mathbb{R}^{n_m})$ to $wL^{Q}(\mathbb{R}^{n_1}\times\cdots\times\mathbb{R}^{n_m})$.

Theorem 2.1 Let $0<\beta_i<n_i$, $\boldsymbol{Q}=\left(\dfrac{n_1}{n_1-\beta_2},\cdots,\dfrac{n_m}{n_m-\beta_m}\right)$, $i=1,\cdots,m$. Then the operator $\mathbb{H}_{\beta_1,\cdots,\beta_n}$ defined by (2) is bounded from $L^1(\mathbb{R}^{n_1}\times\cdots\times\mathbb{R}^{n_m})$ to $wL^{Q}(\mathbb{R}^{n_1}\times\cdots\times\mathbb{R}^{n_m})$. Furthermore,

$$\|\mathbb{H}_{\beta_1,\cdots,\beta_m}\|_{L^1(\mathbb{R}^{n_1}\times\cdots\times\mathbb{R}^{n_m})\to wL^{Q}(\mathbb{R}^{n_1}\times\cdots\times\mathbb{R}^{n_m})}=1. \tag{3}$$

Proof To make the arguments more easily understood, we prove the boundedness of the case $m=2$ first, and then the case $m\geqslant 3$ is just a repetition of the case $m=2$.

For $m=2$, the operator $\mathbb{H}_{\beta_1,\beta_2}$ can be written as

$$(\mathbb{H}_{\beta_1,\beta_2}f)(x_1,x_2)=\frac{1}{|B(0,|x_1|)|^{1-\frac{\beta_1}{n_1}}}\frac{1}{|B(0,|x_2|)|^{1-\frac{\beta_2}{n_2}}}\int_{|y_1|<|x_1|}\int_{|y_2|<|x_2|}f(y_1,y_2)\,\mathrm{d}y_1\mathrm{d}y_2.$$

When $f\in L^1(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$, we have

$$\frac{1}{|B(0,|x_2|)|^{1-\frac{\beta_2}{n_2}}}\int_{|y_2|<|x_2|}f(\cdot,y_2)\,\mathrm{d}y_2\in L^1(\mathbb{R}^{n_1})$$

for all $x_2\in\mathbb{R}^{n_2}$. Then Theorem A yields that

$$\|(\mathbb{H}_{\beta_1,\beta_2}f)(\cdot,x_2)\|_{L^{\frac{n_1}{n_1-\beta_1},\infty}(R^{n_1})}=\sup_{\lambda_1>0}\lambda_1\left|\{x_1:(\mathbb{H}_{\beta_1,\beta_2}f)(x_1,x_2)>\lambda_1\}\right|^{\frac{n_1}{n_1-\beta_1}}\leqslant 1\cdot\left\|\frac{1}{|B(0,|x_2|)|^{1-\frac{\beta_2}{n_2}}}\int_{|y_2|<|x_2|}f(\cdot,y_2)\,\mathrm{d}y_2\right\|_{L^1(\mathbb{R}^{n_1})}. \tag{4}$$

Using Lemma 1.1 (or Fubini's theorem), it is not hard for us to get

$$\begin{aligned}&\left\|\frac{1}{|B(0,|x_2|)|^{1-\frac{\beta_2}{n_2}}}\int_{|y_2|<|x_2|}f(\cdot,y_2)\,\mathrm{d}y_2\right\|_{L^1(\mathbb{R}^{n_1})}\\&=\frac{1}{|B(0,|x_2|)|^{1-\frac{\beta_2}{n_2}}}\int_{\mathbb{R}^{n_1}}\left|\int_{|y_2|<|x_2|}f(y_1,y_2)\,\mathrm{d}y_2\right|\mathrm{d}y_1\\&\leqslant\frac{1}{|B(0,|x_2|)|^{1-\frac{\beta_2}{n_2}}}\int_{|y_2|<|x_2|}\int_{\mathbb{R}^{n_1}}|f(y_1,y_2)|\,\mathrm{d}y_1\mathrm{d}y_2.\end{aligned} \tag{5}$$

Obviously, $\int_{\mathbb{R}^{n_1}}|f(y_1,y_2)|\,\mathrm{d}y_1\in L^1(\mathbb{R}^{n_2})$ if $f\in L^1(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$. Then, applying Lemma 1.1 again, we obtain

$$\begin{aligned}&\left\|\frac{1}{|B(0,|x_2|)|^{1-\frac{\beta_2}{n_2}}}\int_{|y_2|<|x_2|}\left(\int_{\mathbb{R}^{n_1}}|f(y_1,y_2)|\,\mathrm{d}y_1\right)\mathrm{d}y_2\right\|_{L^{\frac{n_2}{n_2-\beta_2},\infty}(\mathbb{R}^{n_2})}\\&=\sup_{\lambda_2>0}\lambda_2\left|\left\{x_2:\frac{1}{|B(0,|x_2|)|^{1-\frac{\beta_2}{n_2}}}\int_{|y_2|<|x_2|}\left(\int_{\mathbb{R}^{n_1}}|f(y_1,y_2)|\,\mathrm{d}y_1\right)\mathrm{d}y_2>\lambda_2\right\}\right|^{\frac{n_2}{n_2-\beta_2}}\\&\leqslant 1\cdot\int_{\mathbb{R}^{n_2}}\int_{\mathbb{R}^{n_1}}|f(y_1,y_2)|\,\mathrm{d}y_1\mathrm{d}y_2\\&=\|f\|_{L^1(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}.\end{aligned} \tag{6}$$

Combining (4), (5), and (6), we obtain

$$\|\mathbb{H}_{\beta_1,\beta_2}f\|_{wL^{\frac{n_1}{n_1-\beta_2},\frac{n_2}{n_2-\beta_2}}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}\leqslant 1\cdot\|f\|_{L^1(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})},$$

for all $f\in L^1(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$.

On the other hand, we will show that the constant 1 is the best possible. Denote $\chi_{[0,1]}$ by χ_1. Taking $f_0(r)=\chi_1(r)$, $r>0$ and choosing $F(x_1,x_2)=f_0(|x_1|)f_0(|x_2|)$, where $x_1\in\mathbb{R}^{n_1}$, $x_2\in\mathbb{R}^{n_2}$, we get from the definition of $\mathbb{H}_{\beta_1,\beta_2}$ that

$$\begin{aligned}\mathbb{H}_{\beta_1,\beta_2}F(x_1,x_2)&=\frac{1}{|B(0,|x_1|)|^{1-\frac{\beta_1}{n_1}}}\frac{1}{|B(0,|x_2|)|^{1-\frac{\beta_2}{n_2}}}\int_{|y_1|<|x_1|}\int_{|y_2|<|x_2|}f_0(|y_1|)f_0(|y_2|)\,\mathrm{d}y_1\mathrm{d}y_2\\&=\frac{1}{|B(0,|x_1|)|^{1-\frac{\beta_1}{n_1}}}\int_{|y_1|<|x_1|}f_0(|y_1|)\,\mathrm{d}y_1\\&\quad\frac{1}{|B(0,|x_2|)|^{1-\frac{\beta_2}{n_2}}}\int_{|y_2|<|x_2|}f_0(|y_2|)\,\mathrm{d}y_2\end{aligned}$$

$$= \mathbb{H}_{\beta_1} f_0(|x_1|)\, \mathbb{H}_{\beta_2} f_0(|x_2|).$$

For $0 < \lambda_1 < \frac{1}{|B(0,|x_2|)|^{1-\frac{\beta_2}{n_2}}} \int_{|y_2|<|x_2|} f_0(|y_2|)\mathrm{d}y_2 = \mathbb{H}_{\beta_2} f_0(|x_2|)$, we divide x_1 into two cases:

(i) when $|x_1| < 1$,

$$|\mathbb{H}_{\beta_1} f_0(|x_1|)| = \frac{1}{|B(0,|x_1|)|^{1-\frac{\beta_1}{n_1}}} \int_{|y_1|<|x_1|} f_0(|y_1|)\mathrm{d}y_1 = \frac{1}{|B(0,|x_1|)|^{1-\frac{\beta_1}{n_1}}} \int_{|y_1|<|x_1|} \chi_1(|y_1|)\mathrm{d}y_1 = \frac{v_{n_1}|x_1|^{n_1}}{v_{n_1}^{1-\frac{\beta_1}{n_1}}|x_1|^{n_1-\beta_1}};$$

(ii) when $|x_1| \geqslant 1$,

$$|\mathbb{H}_{\beta_1} f_0(|x_1|)| = \frac{1}{|B(0,|x_1|)|^{1-\frac{\beta_1}{n_1}}} \int_{|y_1|<|x_1|} \chi_1(|y_1|)\mathrm{d}y_1 = \frac{1}{|B(0,|x_1|)|^{1-\frac{\beta_1}{n_1}}} \int_{|y_1|<1} \mathrm{d}y_1 = \frac{v_{n_1}}{v_{n_1}^{1-\frac{\beta_1}{n_1}}|x_1|^{n_1-\beta_1}}.$$

Then, combining both the cases we obtain

$$\begin{aligned} &|\{x_1 \in \mathbb{R}^{n_1}: |\mathbb{H}_{\beta_1,\beta_2} F(x_1,x_2)| > \lambda_1\}| = \\ &\quad |B(0,1) \cap \{x_1 \in \mathbb{R}^{n_1}: v_{n_1}^{\frac{\beta_1}{n_1}} |x_1|^{\beta_1} \mathbb{H}_{\beta_2} f_0(|x_2|) > \lambda_1\}| + \\ &\quad |B(0,1)^c \cap \{x_1 \in \mathbb{R}^{n_1}: v_{n_1}^{\frac{\beta_1}{n_1}} |x_1|^{\beta_1-n_1} \mathbb{H}_{\beta_2} f_0(|x_2|) > \lambda_1\}| \\ &= \left|\left\{x_1 \in \mathbb{R}^{n_1}: \frac{\lambda_1^{\frac{1}{\beta_1}}}{v_{n_1}^{\frac{1}{n_1}} \mathbb{H}_{\beta_2}^{\frac{1}{\beta_1}} f_0(|x_2|)} < |x_1| < 1\right\}\right| + \\ &\quad \left|\left\{x_1 \in \mathbb{R}^{n_1}: 1 \leqslant |x_1| < \left(\frac{v_{n_1}^{\frac{\beta_1}{n_1}} \mathbb{H}_{\beta_2} f_0(|x_2|)}{\lambda_1}\right)^{\frac{1}{n_1-\beta_1}}\right\}\right| \\ &= \left(\frac{v_{n_1} \mathbb{H}_{\beta_2} f_0(|x_2|)}{\lambda_1}\right)^{\frac{n_1}{n_1-\beta_1}} - \left(\frac{\lambda_1}{\mathbb{H}_{\beta_2} f_0(|x_2|)}\right)^{\frac{n_1}{\beta_1}}. \end{aligned}$$

Based on the above results, let $\lambda_1 \to 0^+$ and we have the estimate

$$\lim_{\lambda_1 \to 0} \lambda_1 |\{x_1 \in \mathbb{R}: |\mathbb{H}_{\beta_1,\beta_2} F(x_1,x_2)| > \lambda_1\}|^{\frac{n_1-\beta_1}{n_1}} = v_{n_1} \mathbb{H}_{\beta_2} f_0(|x_2|), \tag{7}$$

which implies that

$$\sup_{0<\lambda_1<\mathbb{H}_{\beta_2} f_0(|x_2|)} \lambda_1 |\{x_1: |\mathbb{H}_{\beta_1,\beta_2} F(x_1,x_2)| > \lambda_1\}|^{\frac{n_1}{n_1-\beta_1}} = v_{n_1} \mathbb{H}_{\beta_2} f_0(|x_2|). \tag{8}$$

For $0 < \lambda_2 < v_{n_1}$, we also divide x_2 into two cases: $|x_2| < 1$ and $|x_2| \geqslant 1$. As above, we obtain

$$\sup_{0<\lambda_2<v_{n_1}} \lambda_2 |\{x_2 \in \mathbb{R}^{n_2}: v_{n_1} \mathbb{H}_{\beta_2} f_0(|x_2|) > \lambda_2\}|^{\frac{n_2}{n_2-\beta_2}} = v_{n_1} v_{n_2}. \tag{9}$$

Since $\|F\|_{L^1(\mathbb{R}^{n_1 \times n_2})} = v_{n_1} v_{n_2}$, by combining (8) with (9) we obtain

$$\|\mathbb{H}_{\beta_1,\beta_2} F\|_{wL^{\frac{n_1}{n_1-\beta_1},\frac{n_2}{n_2-\beta_2}}(\mathbb{R}^{n_1 \times n_2})} = 1 \cdot \|F\|_{L^1(\mathbb{R}^{n_1 \times n_2})}.$$

This finishes the proof of Theorem 2.1. □

At last, we prove a more general result involved the weak and strong mixed-norm space.

Theorem 2.2 Suppose $\boldsymbol{P} = (p_1, \cdots, p_m) \geqslant \boldsymbol{1}$, $\boldsymbol{Q} = (q_1, \cdots, q_m) \geqslant \boldsymbol{1}$, and for all $i = 1, \cdots, m$, $0 < \beta_i < n$, $\frac{1}{p_i} - \frac{1}{q_i} = \frac{\beta_i}{n_i}$. Let $I \subset \{1, \cdots, m\}$, and we further suppose $1 < p_i < \infty$ if $i \in I$. Then the operator $\mathbb{H}_{\beta_1,\cdots,\beta_m}$ defined by (2) is bounded from $L^{\boldsymbol{P}}(\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_m})$ to $L^{\boldsymbol{Q}_I}(\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_m})$. Moreover, there holds

$$\|\mathbb{H}_{\beta_1,\cdots,\beta_m}\|_{L^{\boldsymbol{P}}(\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_m}) \to L^{\boldsymbol{Q}_I}(\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_m})} = \prod_{i \in I} C_i, \tag{10}$$

where

$$C_i = \left(\frac{p_i'}{q_i}\right)^{\frac{1}{q_i}} \left(\frac{n_i}{q_i \beta_i} \cdot B\left(\frac{n_i}{q_i \beta_i}, \frac{n_i}{q_i' \beta_i}\right)\right)^{-\frac{\beta_i}{n_i}}.$$

Proof We only prove the case $m = 2$ and $I = \{1\}$ for simplification, that is to say, for $p_1 > 1$ and $p_2 \geqslant 1$ we need to verify

$$\|\mathbb{H}_{\beta_1,\beta_2} f\|_{L^{\boldsymbol{Q}_I}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} \leqslant C_1 \cdot \|f\|_{L^{p_1,p_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})}, \tag{11}$$

where the constant C_1 is sharp.

Noting that, when $f \in L^{p_1,p_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$,

$$\frac{1}{|B(0,|x_2|)|^{1-\frac{\beta_2}{n_2}}} \int_{|y_2|<|x_2|} f(\cdot, y_2)\mathrm{d}y_2 \in L^{p_1}(\mathbb{R}^{n_1}),$$

for all $x_2 \in \mathbb{R}^{n_2}$, we get from Theorem B that

$$\|(\mathbb{H}_{\beta_1,\beta_2} f)(\cdot, x_2)\|_{L^{q_1}(\mathbb{R}^{n_1})} \leqslant C_1 \times \left\|\frac{1}{|B(0,|x_2|)|^{1-\frac{\beta_2}{n_2}}} \int_{|y_2|<|x_2|} f(\cdot, y_2)\mathrm{d}y_2\right\|_{L^{p_1}(\mathbb{R}^{n_1})}. \tag{12}$$

Using Lemma 1.1 (or Fubini's theorem), it is not hard for us to get

$$\left\| \frac{1}{|B(0,|x_2|)|^{1-\frac{\beta_2}{n_2}}} \int_{|y_2|<|x_2|} f(\cdot, y_2)\,\mathrm{d}y_2 \right\|_{L^{p_1}(\mathbb{R}^{n_1})}$$

$$= \frac{1}{|B(0,|x_2|)|^{1-\frac{\beta_2}{n_2}}} \left[\int_{\mathbb{R}^{n_1}} \left| \int_{|y_2|<|x_2|} f(y_1,y_2)\,\mathrm{d}y_2 \right|^{p_1} \mathrm{d}y_1 \right]^{\frac{1}{p_1}}$$

$$\leqslant \frac{1}{|B(0,|x_2|)|^{1-\frac{\beta_2}{n_2}}} \int_{|y_2|<|x_2|} \left[\int_{\mathbb{R}^{n_1}} |f(y_1,y_2)|^{p_1} \mathrm{d}y_1 \right]^{\frac{1}{p_1}} \mathrm{d}y_2.$$

Theorem B tells us that, when $f \in L^{p_1,p_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$, there holds

$$\left\| \left\| \frac{1}{|B(0,|x_2|)|^{1-\frac{\beta_2}{n_2}}} \int_{|y_2|<|x_2|} \left[\int_{\mathbb{R}^{n_1}} |f(y_1,y_2)|^{p_1} \mathrm{d}y_1 \right]^{\frac{1}{p_1}} \mathrm{d}y_2 \right\|_{L^{p_1}(\mathbb{R}^{n_1})} \right\|_{L^{q_2,\infty}(\mathbb{R}^{n_2})} \leqslant 1 \cdot \|f\|_{L^{p_1,p_2}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}.$$

Based on the above estimates, we immediately obtain the inequality (11).

In order to show that C_1 is the best, we construct a sharp function

$$F(x_1,x_2) = f_1(x_1) f_2(x_2),$$

where

$$f_1(x_1) = \frac{1}{(1+|x_1|^{q_1\beta_1})^{1+\frac{n_1}{q_1\beta_1}}},$$

and

$$f_2(x_2) = \chi_1(|x_2|).$$

As stated in Theorem B, it means

$$\frac{\|\mathbb{H}_{\beta_1} f_1\|_{L^{p_1}(\mathbb{R}^{n_1})}}{\|f_1\|_{L^{p_1}(\mathbb{R}^{n_1})}} = C_1. \tag{13}$$

From the proof of Theorem A, we have

$$\frac{\|\mathbb{H}_{\beta_2} f_2\|_{wL^{p_2}(\mathbb{R}^{n_2})}}{\|f_2\|_{L^{p_2}(\mathbb{R}^{n_2})}} = 1. \tag{14}$$

Combining (13) with (14) we obtain

$$\|\mathbb{H}_{\beta_1,\beta_2}\|_{L^{p_1,p_2}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}) \to L^{Q_I}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}$$

$$= \sup_{\|F\|_{L^{p_1,p_2}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})} \neq 0} \frac{\|\mathbb{H}_{\beta_1,\beta_2} F\|_{L^{Q_I}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}}{\|F\|_{L^{p_1,p_2}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}}$$

$$\geqslant \frac{\|\mathbb{H}_{\beta_1,\beta_2} f_1 f_2\|_{L^{Q_I}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}}{\|f_1 f_2\|_{L^{p_1,p_2}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}}$$

$$= \frac{\|\mathbb{H}_{\beta_1} f_1\|_{L^{p_1}(\mathbb{R}^{n_1})}}{\|f_1\|_{L^{p_1}(\mathbb{R}^{n_1})}} \cdot \frac{\|\mathbb{H}_{\beta_2} f_2\|_{wL^{p_2}(\mathbb{R}^{n_2})}}{\|f_2\|_{L^{p_2}(\mathbb{R}^{n_2})}} = C_1.$$

Thus we finish the proof. □

We would like to remark that similar results have been deduced by authors of Ref. [11] for Hardy operator on higher-dimensional product spaces.

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