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# Conformal minimal immersions of $S^2$ into $\mathbb{H}P^4$

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**Abstract** This work is a generalization of Chen and Jiao's work, where they considered the question of explicit construction of some conformal minimal two-spheres of constant curvature in quaternionic projective space. The crucial point was to find some horizontal immersions derived from Veronese sequence in $\mathbb{C}P^{2n+1}$, which was projected into constant curvature conformal minimal two-spheres by twistor map $\pi: \mathbb{C}P^{2n+1} \to \mathbb{H}P^n$. They calculated the case $n=2$. In this work, we deal with the case $n = 4$ and a related geometry phenomenon.

**Keywords** minimal two-sphere; Gaussian curvature; Veronese sequence; quaternionic projective space

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# $S^2$ 到 $\mathbb{H}P^4$ 的共形极小浸入

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**摘 要** 本工作是 Chen 和 Jiao 工作的推广。他们考虑在四元数射影空间中如何具体构造常曲率共形极小二球,关键点是从 $\mathbb{C}P^{2n+1}$ 里的 Veronese 序列找到一些相关的水平浸入,然后关于扭映射 $\pi: \mathbb{C}P^{2n+1} \to \mathbb{H}P^n$ 做投影就得到 $\mathbb{H}P^n$ 的常曲率共形极小二球。Chen 和 Jiao 计算了 $n=2$ 的情况,本工作处理 $n=4$ 的情况和一个相关的几何现象。

**关键词** 极小二球; 高斯曲率; Veronese 序列; 四元数射影空间

There have been many researches about minimal surfaces in real and complex space forms[1-3]. For quaternionic projective space, Bryant[4] proved that the projections of horizontal holomorphic surfaces in $\mathbb{C}P^3$ are minimal in $\mathbb{H}P^1$ through twistor map. Aithal[5] described all the harmonic maps from $S^2$ to $\mathbb{H}P^2$. Considering $\mathbb{H}P^n$ as a totally geodesic submanifold in the complex Grassmann manifold $G(2, 2n+2)$, Bahy-El-Dien and Wood[6] constructed explicitly all harmonic maps from the two-spheres to a quaternionic projective space. Based on Bahy-El-Dien and Wood's results, some classifications of minimal two-spheres of constant curvature in $\mathbb{H}P^n$ have been given[7-9].

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Recently, Chen and Jiao[10] used twistor map to discuss conformal minimal surface in quaternionic projective space, based on the horizontal condition given by Yang[11]. They got two polynomial equations as the horizontal conditions for the immersed surfaces in $\mathbb{C}P^{2n+1}$, and then they calculated Veronese surfaces in $\mathbb{C}P^5$ and got a series of conformal minimal two-spheres with constant curvature in $\mathbb{H}P^2$.

In this paper we deal with the case $\mathbb{H}P^4$.

In section 1, we review the knowledge of geometry of quaternionic projective space, minimal surfaces in complex projective space, and horizontal equation of immersed surfaces with respect to the twistor map.

In section 2, we try to improve the calculation of Veronese two-spheres to the case $n$. We will calculate explicitly the case $n=4$ and get a series of conformal minimal two-spheres $\phi_{4,p} = \boldsymbol{\pi} \circ (\boldsymbol{U}_0 \cdot V_p^9)(0 \leqslant p \leqslant 3)$ in $\mathbb{H}P^4$.

In section 3, we talk about the interesting phenomenon that the middle terms, $V_4^9$ and $V_5^9$, of the Veronese sequences can not be rotated horizontally with respect to the twistor map. We will use the method of harmonic sequence to prove it.

# 1 Preliminaries

## 1.1 Geometry of quaternionic projective space

Let $\mathbb{H}$ be the quaternion algebra, which is regarded as a 4-dimensional real vector space with the basis 1, i, j, k. A ring multiplication is defined as

$$\mathrm{i}^2 = \mathrm{j}^2 = \mathrm{k}^2 = -1,$$
$$\mathrm{ij} = \mathrm{k} = -\mathrm{ji}, \mathrm{jk} = \mathrm{i} = -\mathrm{kj}, \mathrm{ki} = \mathrm{j} = -\mathrm{ik}. \quad (1)$$

In particular, $\mathbb{H}$ is a 2-dimensional ring module over $\mathbb{C}$ with the basis 1, j. For $z = z_1 + \mathrm{j}z_2, w = w_1 + \mathrm{j}w_2 \in \mathbb{H}$, we have

$$zw = (z_1w_1 - \bar{z}_2w_2) + \mathrm{j}(z_2w_1 + \bar{z}_1w_2). \quad (2)$$

There is a natural conjugation in $\mathbb{H}$, denoted by

$$(z_1 + \mathrm{j}z_2)^* = z_1 - \mathrm{j}z_2. \quad (3)$$

Let $\mathbb{H}^n$ be the set of $n$-tuples of quaternions written as column vectors. We can identify $\mathbb{H}^n$ with $\mathbb{C}^{2n}$:

$$z_1 + \mathrm{j}z_2 \mapsto \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}. \quad (4)$$

The quaternionic projective space $\mathbb{H}P^n$ is the space of $\mathbb{H}$-lines in $\mathbb{H}^{n+1}$, that is, for $[v_1]_{\mathbb{H}}$ and $[v_2]_{\mathbb{H}} \in \mathbb{H}P^n$, $[v_1]_{\mathbb{H}} = [v_2]_{\mathbb{H}}$ if and only if there is an $x \in \mathbb{H}$, such that $v_2 = v_1 x$.

The symplectic group is

$$Sp(n) = \{\boldsymbol{A} \in \mathrm{GL}(n;\mathbb{H}) \mid {}^t\boldsymbol{A}^* \cdot \boldsymbol{A} = \boldsymbol{I}_n\}, \quad (5)$$

where $\boldsymbol{I}_n$ is the identity matrix of order $n$.

For $\boldsymbol{A} \in Sp(n+1)$ and $[\boldsymbol{v}]_{\mathbb{H}} \in \mathbb{H}P^n$, $Sp(n+1)$ acts on $\mathbb{H}P^n$ by

$$\boldsymbol{A} \cdot [\boldsymbol{v}]_{\mathbb{H}} = [\boldsymbol{A}\boldsymbol{v}]_{\mathbb{H}}. \quad (6)$$

$G_0 = Sp(1) \times Sp(n)$ is the isotropy group of this action at $[(1, 0, \cdots, 0)^{\mathrm{T}}]_{\mathbb{H}}$ and this action is transitive. Therefore,

$$\mathbb{H}P^n = Sp(n+1)/Sp(1) \times Sp(n). \quad (7)$$

There is the inclusion $Sp(n) \hookrightarrow SU(2n)$ by $\boldsymbol{X} + \mathrm{j}\boldsymbol{Y} \mapsto \begin{pmatrix} \boldsymbol{X} & -\bar{\boldsymbol{Y}} \\ \boldsymbol{Y} & \bar{\boldsymbol{X}} \end{pmatrix}$, where $\boldsymbol{X}, \boldsymbol{Y} \in GL(n;C)$.

We have the following commutative diagram

$$\begin{array}{ccc} & Sp(n+1) & \\ \pi_2 \swarrow & & \searrow \pi_1 \\ \mathbb{H}P^n & \xleftarrow{\pi} & \mathbb{C}P^{2n+1} \end{array} \quad (8)$$

where $\boldsymbol{\pi}: \mathbb{C}P^{2n+1} \to \mathbb{H}P^n, \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \mapsto [z_1 + \mathrm{j}z_2]_{\mathbb{H}}, z_1, z_2 \in \mathbb{C}^{n+1}$, is the twistor map and it is a fibration, $\boldsymbol{\pi}_1$ and $\boldsymbol{\pi}_2$ are defined as follows. If $\boldsymbol{A} = \boldsymbol{X} + \mathrm{j}\boldsymbol{Y} \in Sp(n+1)$, write $\boldsymbol{A} = (A_0, \cdots, A_n)$, $\boldsymbol{X} = (X_0, \cdots, X_n)$, $\boldsymbol{Y} = (Y_0, \cdots, Y_n)$. Then $\boldsymbol{\pi}_1(\boldsymbol{A}) = \begin{bmatrix} X_0 \\ Y_0 \end{bmatrix}$ and $\boldsymbol{\pi}_2(\boldsymbol{A}) = [A_0]_{\mathbb{H}}$ are the natural projections. Note that $Sp(n+1) \to \mathbb{C}P^{2n+1}$ is a principal fiber bundle with structure group $U(1) \times Sp(n)$ and $\boldsymbol{\pi}_1$ is a surjection.

**Definition 1.1** The twistor map gives a fibration, and the horizontal distribution $\mathscr{H}$ on $\mathbb{C}P^{2n+1}$ is defined to be the orthogonal complement to the fiber of $\boldsymbol{\pi}$ with respect to the Fubini-Study metric on $\mathbb{C}P^{2n+1}$. In particular, if $M$ is a Riemann surface, call a map $f: M \to \mathbb{C}P^{2n+1}$ horizontal surface if the image of $f$ is tangent to $\mathscr{H}$.

As in Ref[11], for $[\boldsymbol{v}] \in \mathbb{C}P^{2n+1}$, we write $\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \mathbb{C}^{2n+2}$, where $v_1, v_2 \in \mathbb{C}^{n+1}$, and the horizontal space $\mathscr{H}_{[v]}$ is

$$\mathscr{H}_{[v]} = \{w \in v^{\perp} \mid \sigma_v(w) = 0, \sigma_v = -{}^t z_2 \mathrm{d}z_1 + {}^t z_1 \mathrm{d}z_2\}. \tag{9}$$

We have the following proposition[10], which is a special case of Lemma 3.5 of Eells and Wood[3].

**Proposition 1.1** If $\phi = [f]: M \to \mathbb{C}P^{2n+1}$ is a horizontal conformal minimal surface, then $\pi \circ \phi: M \to \mathbb{H}P^n$ is conformal minimal, where $\pi: \mathbb{C}P^{2n+1} \to \mathbb{H}P^n$ is the twistor map.

## 1.2 Minimal surface in a complex projective space

In this subsection, we introduce some knowledge about harmonic sequences and minimal surfaces in $\mathbb{C}P^n$ [2-3].

Let $\mathbb{C}P^n$ denote the complex lines in $\mathbb{C}^{n+1}$, let $T \to \mathbb{C}P^n$ denote the universal line bundle whose fibre at $l \in \mathbb{C}P^n$ is $l$ itself.

Let $\boldsymbol{M}$ be a Riemann surface. We identify a smooth map $\phi: \boldsymbol{M}^2 \to \mathbb{C}P^n$ with a subbundle $\underline{\phi}$ of the trivial bundle $\underline{\mathbb{C}^{n+1}} = \boldsymbol{M}^2 \times \mathbb{C}^{n+1}$ of rank one which has fibre at $x \in \boldsymbol{M}$ given by $\underline{\phi}_x = \phi(x)$. Thus $\underline{\phi} = \phi^{-1}T$. Conversely any rank subbundle of $\underline{\mathbb{C}^{n+1}}$ induces a map $\boldsymbol{M}^2 \to \mathbb{C}P^n$.

Each linearly full conformal minimal immersion $\varphi$ from $S^2$ is obtained from a holomorphic curve $\varphi_0$ in $\mathbb{C}P^n$, and $\varphi_0$ induces the harmonic sequence,

$$0 \xrightarrow{\partial} \underline{\varphi_0^n} \xrightarrow{\partial} \cdots \xrightarrow{\partial} \underline{\varphi} = \underline{\varphi_\alpha^n} \xrightarrow{\partial} \cdots \xrightarrow{\partial} \underline{\varphi_n^n} \xrightarrow{\partial} 0. \tag{10}$$

Under local coordinate $z$, choose a local holomorphic section $f_0$ of $\underline{\varphi_0^n}$, such that $\frac{\partial}{\partial \bar{z}} f_0 = 0$. We define local section $f_{i+1}$ of $\underline{\varphi_{i+1}^n}$ by the formula

$$f_{i+1} = \frac{\partial}{\partial z} f_i - \frac{\left\langle \frac{\partial}{\partial z} f_i, f_i \right\rangle}{|f_i|^2} f_i, 0 \leqslant i \leqslant n-1. \tag{11}$$

Let $\gamma_i = \frac{|f_{i+1}|^2}{|f_i|^2}$. Then we have the equations[2]

$$\frac{\partial}{\partial \bar{z}} f_i = -\gamma_{i-1} f_{i-1}, 1 \leqslant i \leqslant n,$$
$$\frac{\partial^2}{\partial z \partial \bar{z}} \log |f_j|^2 = \gamma_j - \gamma_{j-1},$$
$$\frac{\partial^2}{\partial z \partial \bar{z}} \log \gamma_j = \gamma_{j+1} - 2\gamma_j + \gamma_{j-1}, 0 \leqslant j \leqslant n-1. \tag{12}$$

For convenience, we set $\gamma_{-1} = \gamma_n = 0$. Bolton et al. [12] defined the Veronese sequence as follows: Let $V_p^n: S^2 \to \mathbb{C}P^n$ be given by

$$V_p^n(z) = [{}^t(g_{p,0}(z), \cdots, g_{p,n}(z))], \tag{13}$$

where $z$ is a local holomorphic coordinate of $S^2$, $p = 0, 1, \cdots, n$. For $j = 0, 1, \cdots, n$, $g_{p,j}$ are given by

$$g_{p,j}(z) = \frac{p!}{(1 + z\bar{z})^p} \sqrt{C_n^j} z^{j-p} \sum_k (-1)^k C_j^{p-k} C_{n-j}^k (z\bar{z})^k. \tag{14}$$

Then, for each $p$, the Veronese map (or Veronese surface) $V_p^n$ is a conformal minimal immersion with constant curvature $\frac{4}{n + 2p(n-p)}$, and $\{V_0^n, \cdots, V_n^n\}$ is the Veronese sequence in $\mathbb{C}P^n$.

**Remark 1.1** Notice that $g_{p,j}(z)$ have a common factor $\frac{p!}{(1 + z\bar{z})^p}$. We can omit it in the horizontal equations which are mentioned below.

The Veronese sequence is particularly important for the following rigidity theorem[2].

**Lemma 1.1** Let $\boldsymbol{\Psi}: S^2 \to \mathbb{C}P^n$ be a linearly full conformal minimal immersion with constant curvature, then, up to a holomorphic isometry of $\mathbb{C}P^n$, the immersion $\boldsymbol{\Psi}$ is an element of the Veronese sequence.

## 1.3 Horizontal equations of immersed surfaces

Based on the above discussions, we can produce conformal minimal two-spheres from the Veronese two-spheres in $\mathbb{H}P^n$ by a unitary rotation. It is stated in the following proposition[10].

**Proposition 1.2** Let $\boldsymbol{\Psi}: S^2 \to \mathbb{C}P^{2n+1}$ be an element of the Veronese sequence in $\mathbb{C}P^{2n+1}$. If there exists a $\boldsymbol{U} \in \boldsymbol{U}(2n+2)$ such that $\boldsymbol{U} \cdot \boldsymbol{\Psi}$ is horizontal in $\mathbb{C}P^{2n+1}$, then $\pi \circ (\boldsymbol{U} \cdot \boldsymbol{\Psi}): S^2 \to \mathbb{H}P^n$ is a conformal minimal two-spheres with constant curvature, where $\pi: \mathbb{C}P^{2n+1} \to \mathbb{H}P^n$ is the twistor map.

From the horizontal condition, if $\boldsymbol{\Psi} = [f]$ is horizontal, then ${}^t f \boldsymbol{B} \mathrm{d} f = 0$, where $\boldsymbol{B} = \begin{pmatrix} 0 & \boldsymbol{I} \\ -\boldsymbol{I} & 0 \end{pmatrix}$. More explictly, it is the following proposition[10].

**Proposition 1.3** For $\boldsymbol{\Psi} = [f] = [{}^t(f_1, f_2, \cdots, f_{2n+2})]: S^2 \to \mathbb{C}P^{2n+1}$, if $\boldsymbol{U} \in \boldsymbol{U}(2n+2)$, then $\boldsymbol{U} \cdot \boldsymbol{\Psi}$ is horizontal if and only if

$$\begin{cases} \sum_{i,j=1}^{2n+2} A_{ij} f_i \partial f_j = 0, \\ \sum_{i,j=1}^{2n+2} A_{ij} f_i \bar{\partial} f_j = 0, \end{cases} \tag{15}$$

where $\boldsymbol{A} = (A_{ij}) = {}^t\boldsymbol{U}\boldsymbol{B}\boldsymbol{U}$, $\partial = \dfrac{\partial}{\partial z}$, $\bar{\partial} = \dfrac{\partial}{\partial \bar{z}}$, and $z$ is a local coordinate of $S^2$.

## 2 Conformal minimal two-spheres in $\mathbb{H}P^4$

### 2.1 Reduced horizontal equation of Veronese maps

**Definition 2.1** Let $\boldsymbol{M}$ be a Riemann surface, we call two surfaces $\boldsymbol{\Psi}, \boldsymbol{\Phi}: \boldsymbol{M}^2 \to \mathbb{H}P^n$ are symplectic equivalent if there is an isometry, i. e., a symplectic matrix $\boldsymbol{H}$, $\boldsymbol{H} \cdot \boldsymbol{\Psi} = \boldsymbol{\Phi}$.

For Veronese sequence, $V_p^{2n+1}$ and $V_{2n+1-p}^{2n+1}$ have the following relation.

**Lemma 2.1** $V_{2n+1-p}^{2n+1} = J_{2n+2} \cdot \bar{V}_p^{2n+1}$, here we choose $V_p^{2n+1} = [{}^t(g_{p,0}, \cdots, g_{p,2n+1})]$, where

$$g_{p,j} = \sqrt{C_{2n+1}^j} z^{j-p} \sum_k (-1)^k C_j^{p-k} C_{2n+1-j}^k (zz)^k, \tag{16}$$

$$\boldsymbol{J}_m = \begin{pmatrix} 0 & 0 & \cdots & 0 & (-1)^{m-1} \\ 0 & 0 & \cdots & (-1)^m & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & (-1)^1 & \cdots & 0 & 0 \\ (-1)^0 & 0 & \cdots & 0 & 0 \end{pmatrix}. \tag{17}$$

for any integer $m$.

**Proof**

$$\begin{aligned} & g_{2n+1-p,2n+1-j} \\ & = \sqrt{C_{2n+1}^{2n+1-j}} z^{p-j} \sum_l (-1)^l C_{2n+1-j}^{2n+1-p-l} C_j^l (zz)^l. \end{aligned} \tag{18}$$

Set $k = p + l - j$, then we easily see

$$\begin{aligned} g_{2n+1-p,2n+1-j} &= (-1)^{j-p} \bar{g}_{p,j} \\ &= (-1)^p (-1)^j \bar{g}_{p,j}. \end{aligned} \tag{19}$$

Omit $(-1)^p$, then we get the conclusion. □

Now, if $V_p^{2n+1}$ can be rotated horizontally, $\boldsymbol{U} \cdot V_p^{2n+1}$ satisfies the horizontal equations,

$$\begin{cases} {}^t V_p^{2n+1} \, {}^t\boldsymbol{U}\boldsymbol{B}\boldsymbol{U} V_{p+1}^{2n+1} = 0, \\ {}^t V_p^{2n+1} \, {}^t\boldsymbol{U}\boldsymbol{B}\boldsymbol{U} V_{p-1}^{2n+1} = 0. \end{cases} \tag{20}$$

By complex conjugate, we have

$$\begin{cases} {}^t V_{2n+1-p}^{2n+1} \, {}^t J_{2n+2} \, {}^t\overline{\boldsymbol{U}}\boldsymbol{B}\overline{\boldsymbol{U}} J_{2n+2} V_{2n-p}^{2n+1} = 0, \\ {}^t V_{2n+1-p}^{2n+1} \, {}^t J_{2n+2} \, {}^t\overline{\boldsymbol{U}}\boldsymbol{B}\overline{\boldsymbol{U}} J_{2n+2} V_{2n+2-p}^{2n+1} = 0. \end{cases} \tag{21}$$

That is to say, $\boldsymbol{U} \cdot V_p^{2n+1}$ is horizontal $\Leftrightarrow$ $\overline{\boldsymbol{U}} J_{2n+2} \cdot V_{2n+1-p}^{2n+1}$ is horizontal.

So we only need to consider the case $p \leqslant n$.

Denote $g_{p,j} (0 \leqslant p, j \leqslant n)$ by

$$g_{p,j} = \sqrt{C_n^j} z^{j-p} \sum_k (-1)^k C_j^{p-k} C_{n-j}^k (z\bar{z})^k. \tag{22}$$

By derivation, we get

$$\partial g_{p,j} = \sqrt{C_n^j} z^{j-p-1} \sum_k (-1)^k C_j^{p-k} C_{n-j}^k (j-p+k)(z\bar{z})^k, \tag{23}$$

$$\partial g_{p,j} = \sqrt{C_n^j} z^{j-p+1} \sum_k (-1)^k C_j^{p-k} C_{n-j}^k k (zz)^{k-1}. \tag{24}$$

Set $S_{ij} = A_{ij} \sqrt{C_n^i C_n^j}$, then we have

$$\begin{cases} \sum_{i+j=m} S_{ij} \sum_{k+l=t} C_i^{p-k} C_{n-i}^k C_j^{p-l} C_{n-j}^l (j-p+l) = 0, \\ \sum_{i+j=m} S_{ij} \sum_{k+l=t} C_i^{p-k} C_{n-i}^k C_j^{p-l} C_{n-j}^l l = 0, \end{cases} \tag{25}$$

for any nonnegative integers $m, t$.

If we change the positions of $k$ and $l$ in the second formula (notice that $k + l = t$ = constant for some $t$ and $S_{ij} = - S_{ji}$), then we get

$$\sum_{i+j=m} S_{ij} \sum_{k+l=t} C_i^{p-k} C_{n-i}^k C_j^{p-l} C_{n-j}^l = 0. \tag{26}$$

Then, using $i + j = m$ = constant for some $m$ in the first equation, we finally get the equations

$$\begin{cases} \sum_{i+j=m} S_{ij} \sum_{k+l=t} C_i^{p-k} C_{n-i}^k C_j^{p-l} C_{n-j}^l = 0, \\ \sum_{i+j=m} S_{ij} \sum_{k+l=t} C_i^{p-k} C_{n-i}^k C_j^{p-l} C_{n-j}^l i = 0. \end{cases} \tag{27}$$

Considering the projections of linearly full Veronese maps in $\mathbb{C}P^{2n+1}$, we only need to change $n$ in the above equations into $2n+1$, and then we have the following theorem.

**Theorem 2.1** Let $V_p^{2n+1}: S^2 \to \mathbb{C}P^{2n+1}$ be the Veronese map defined in (13) and (14). If $\boldsymbol{U} \in \boldsymbol{U}(2n+2)$, $\boldsymbol{U} \cdot V_p^{2n+1}$ is horizontal with respect to the twistor map $\boldsymbol{\pi}: \mathbb{C}P^{2n+1} \to \mathbb{H}P^n$, then it satisfies

$$\begin{cases}\sum_{i+j=m} S_{ij} \sum_{k+l=t} C_i^{p-k} C_{2n+1-i}^k C_j^{p-l} C_{2n+1-j}^l = 0, \\ \sum_{i+j=m} S_{ij} \sum_{k+l=t} C_i^{p-k} C_{2n+1-i}^k C_j^{p-l} C_{2n+1-j}^l i = 0,\end{cases} \tag{28}$$

where $\boldsymbol{S} = \boldsymbol{QAQ}$, $\boldsymbol{Q} = \mathrm{diag}\{1,\cdots,\sqrt{C_{2n+1}^i},\cdots,1\}$, $\boldsymbol{A} = {}^t\boldsymbol{UBU}$ is an anti-symmetric unitary matrix. The above equations should be right for any constant nonnegative integers $m \in \{0,\cdots,4n+2\}$, $t \in \{0,\cdots,2p\}$.

## 2.2 Projections of $V_p^9$, $p=0,1,2,3,4$

As an application of (28), in this subsection, we calculate the Veronese maps $V_p^9$, $p=0,1,2,3,4$.

First, we talk about the general $V_0^{2n+1}$, the equations (28) reduce to:

$$\begin{cases}\sum_{i+j=m} S_{ij} = 0, \\ \sum_{i+j=m} S_{ij} i = 0.\end{cases} \tag{29}$$

Notice that $\boldsymbol{A}$ is an anti-symmetric unitary matrix, so $\boldsymbol{S}$ is anti-symmetric too, the first equation is naturally satisfied, and we can reformulate the second to

$$\sum_{i+j=m,j>i} S_{ij}(j-i) = 0, \tag{30}$$

or, write

$$\sum_{i+j=m,j>i} A_{ij}\sqrt{C_{2n+1}^i C_{2n+1}^j}\,(j-i) = 0. \tag{31}$$

It is a linear combination of matrix elements in different lines of upper triangulation part of $\boldsymbol{S}$. Now, we talk about the solution in different $n(n \geqslant 1)$.

First, when $n=1$, the number of elements in the lines which are parallel to antidiagonal in the upper triangulation part of $\boldsymbol{S}$ (respect $\boldsymbol{A}$) is less than 2, then they should be all zeros, so with respect to the twistor map $\pi: \mathbb{C}P^3 \to \mathbb{H}P^1$, there only have the linear relation in the antidiagonal:

$$\sum_{i+j=3,j>i} S_{ij}(j-i) = 0, \tag{32}$$

or, write

$$\sum_{i+j=3,j>i} A_{ij} C_3^i (j-i) = 0. \tag{33}$$

The solution is

$$\begin{pmatrix} 0 & 0 & 0 & A_{03} \\ 0 & 0 & -A_{03} & 0 \\ 0 & A_{03} & 0 & 0 \\ -A_{03} & 0 & 0 & 0 \end{pmatrix} \tag{34}$$

So, when $n=1$, $\boldsymbol{A} = \lambda \boldsymbol{J}_4$, $\lambda \in \mathbb{C}$, $|\lambda| = 1$.

In general, we have

**Lemma 2.2** $\sum_{i+j=2n+1,j>i} A_{ij} C_{2n+1}^i (j-i) = 0$ or $\sum_{i+j=2n+1} A_{ij} C_{2n+1}^i i = 0$ has a solution:

$$\begin{cases} A_{ij} = 0, i+j \neq 2n+1, \\ A_{ij} = (-1)^i A_{0,2n+1}, i+j = 2n+1. \end{cases} \tag{35}$$

or, equivalently, $\boldsymbol{A} = \lambda \boldsymbol{J}_{2n+2}$, $\lambda \in \mathbb{C}$, $|\lambda| = 1$.

**Proof** For $(1+x)^{2n+1} = \sum_{i=0}^{2n+1} C_{2n+1}^i x^i$, derivative both sides. □

**Remark 2.1** Readers could see $\sum_{i+j=2n+1} A_{ij} C_{2n+1}^i i^t = 0$ also have this solution for any nonnegative integer $t$.

**Lemma 2.3** Choose two solutions of ${}^t\boldsymbol{UBU} = \boldsymbol{A}$, denote them by $\boldsymbol{M}, \boldsymbol{N}$, then $\boldsymbol{\pi} \circ (\boldsymbol{M} \cdot V_p^{2n+1})$ is symplectic equivalent to $\boldsymbol{\pi} \circ (\boldsymbol{N} \cdot V_p^{2n+1})$.

**Proof** It's easy to see $\boldsymbol{MN}^{-1}$ are symplectic matrix with the form $\boldsymbol{MN}^{-1} = \begin{pmatrix} \boldsymbol{X} & -\bar{\boldsymbol{Y}} \\ \boldsymbol{Y} & \bar{\boldsymbol{X}} \end{pmatrix}$, denote $\boldsymbol{N} \cdot V_p^{2n+1}$ by $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, then $\boldsymbol{\pi} \circ (\boldsymbol{M} \cdot V_p^{2n+1}) = [\boldsymbol{X}v_1 - \bar{\boldsymbol{Y}}v_2 + \mathrm{j}(\boldsymbol{Y}v_1 + \bar{\boldsymbol{X}}v_2)]_{\mathbb{H}} = (\boldsymbol{X} + \mathrm{j}\boldsymbol{Y}) \cdot [v_1 + \mathrm{j}v_2]_{\mathbb{H}} = (\boldsymbol{X} + \mathrm{j}\boldsymbol{Y}) \cdot (\boldsymbol{\pi} \circ (\boldsymbol{N} \cdot V_p^{2n+1}))$. □

From the above Lemma, the key point of "rotate horizontally" is to decide the existence of $\boldsymbol{A}$, about $\boldsymbol{U}$, we only need to search for the solution of special form, here we consider that $\boldsymbol{U}$ possess this form: $\boldsymbol{U} = \begin{pmatrix} \boldsymbol{U}_1 & 0 \\ 0 & \boldsymbol{I} \end{pmatrix}$, $\boldsymbol{U}_1 \in \boldsymbol{U}(n+1)$.

Then $\boldsymbol{A} = \begin{pmatrix} 0 & {}^t\boldsymbol{U}_1 \\ -\boldsymbol{U}_1 & 0 \end{pmatrix}$, so, when $p=0$, for any $n \geqslant 1$, $\boldsymbol{U}$ has one solution $\boldsymbol{U} = \boldsymbol{U}_0 = \begin{pmatrix} \boldsymbol{J}_{n+1} & 0 \\ 0 & \boldsymbol{I} \end{pmatrix}$.

In the rest part of this article, we will always denote the above matrix by $\boldsymbol{U}_0$.

This solution is special, now $\boldsymbol{U}_0 V_0^{2n+1}$ is horizontal, since $\boldsymbol{U}_0 \boldsymbol{J}_{2n+2} = \begin{pmatrix} 0 & -\boldsymbol{I} \\ \boldsymbol{J}_{n+1} & 0 \end{pmatrix}$, then

$$\boldsymbol{U}_0 V_{2n+1}^{2n+1} = \boldsymbol{U}_0 \boldsymbol{J}_{2n+2} \bar{V}_0^{2n+1} = \mathbf{J} \circ \boldsymbol{U}_0 V_0^{2n+1} \tag{36}$$

is also horizontal.

From the above discussion, we have

**Theorem 2.2** $V_0^9$ can be rotated horizontally, and $\pi \circ (U_0 \cdot V_0^9)$ is conformal minimal two-sphere with constant curvature 4/9 in $\mathbb{H}P^4$.

For $V_1^9$, If we set $F_t = \sum_{i+j=m} S_{ij} i^t$ for some nonnegative integer $m, t$, then the first equation of (28) reduce to

$$\begin{cases} mF_1 - F_2 = 0, \\ 9mF_0 - 2(mF_1 - F_2) = 0, \\ 9(9-m)F_0 + mF_1 - F_2 = 0, \end{cases} \tag{37}$$

the second equation of (28) reduce to

$$\begin{cases} mF_2 - F_3 = 0, \\ 9mF_1 - 2(mF_2 - F_3) = 0, \\ 9(9-m)F_1 + mF_2 - F_3 = 0, \end{cases} \tag{38}$$

notice $F_0 = 0$ is naturally satisfied, then we have

$$F_1 = 0, F_2 = 0, F_3 = 0. \tag{39}$$

The equations $F_t = 0$ which $t$ is even are degenerate to those $t$ is odd, since

$$F_{2k} = 0 \Leftrightarrow \sum_{i+j=m, j>i} S_{ij}((m-i)^{2k} - i^{2k}) = 0, \tag{40}$$

and

$$\sum_{i+j=m, j>i} S_{ij}((m-i)^{2k} - i^{2k}) = 0 \Leftarrow F_t = 0 \tag{41}$$

by every $0 \leqslant t \leqslant 2k-1$.

So, finally, the equations reduce to

$$F_1 = 0, F_3 = 0. \tag{42}$$

The other discussions are just like $V_0^9$, we give the following result:

**Theorem 2.3** For $V_1^9$ can be rotated horizontally, and $\pi \circ (\boldsymbol{U}_0 \cdot V_1^9)$ is conformal minimal two-sphere with constant curvature 4/25 in $\mathbb{H}P^4$.

For $V_2^9$, (28) reduce to

$$\begin{cases} mF_1 - F_2 = 0, \\ mF_2 - F_3 = 0, \\ F_4 - 2mF_3 + m^3 F_1 = 0, \\ F_5 - 2mF_4 + (m^2 + m - 1)F_3 + (m - m^2)F_2 = 0, \\ F_3 - mF_2 + \dfrac{72m(m-1)}{48m - 208} F_1 = 0, \end{cases} \tag{43}$$

then we have

$$F_1 = 0, F_2 = 0, F_3 = 0, F_4 = 0, F_5 = 0. \tag{44}$$

So, finally, the equations reduce to

$$F_1 = 0, F_3 = 0, F_5 = 0. \tag{45}$$

The other discussions are just like $V_0^9$, we give the following result:

**Theorem 2.4** $V_2^9$ can be rotated horizontally, and $\pi \circ (\boldsymbol{U}_0 \cdot V_2^9)$ is conformal minimal two-sphere with constant curvature 4/37 in $\mathbb{H}P^4$.

For $V_3^9$, (28) reduce to

$$\boldsymbol{B}3 * {}^t(F_7, F_6, F_5, F_4, F_3, F_2, F_1, F_0) = 0, \tag{46}$$

where $\boldsymbol{B}3$ is a $14 \times 8$ matrix, it is a big matrix, by using the software Matlab, we calculate rank$(\boldsymbol{B}3) = 8$, so we have

$$F_i = 0 \tag{47}$$

for $0 \leqslant i \leqslant 7$.

The equations $F_t = 0$ which $t$ is even are degenerate to those $t$ is odd.

So, finally, the equations reduce to

$$F_1 = 0, F_3 = 0, F_5 = 0, F_7 = 0. \tag{48}$$

It is still not determinate for the elements of matrix $S$, so we still have a solution $\boldsymbol{U} = \boldsymbol{U}_0$, then:

**Theorem 2.5** $V_3^9$ can be rotated horizontally, and $\pi \circ (\boldsymbol{U}_0 \cdot V_3^9)$ is conformal minimal two-sphere with constant curvature 4/45 in $\mathbb{H}P^4$.

We continue to talk about $V_4^9$, (28) reduce to

$$\boldsymbol{B}4 * {}^t(F_9, F_8, F_7, F_6, F_5, F_4, F_3, F_2, F_1, F_0) = 0, \tag{49}$$

where $\boldsymbol{B}4$ is a $18 \times 10$ matrix, we calculate rank $(\boldsymbol{B}4) = 10$, so we have

$$F_i = 0 \tag{50}$$

for $0 \leqslant i \leqslant 9$.

The equations $F_t = 0$ which $t$ is even are degenerate to those $t$ is odd.

So, finally, the equations reduce to

$$F_1 = 0, F_3 = 0, F_5 = 0, F_7 = 0, F_9 = 0. \tag{51}$$

Different like the former cases, this is a over-determined system for the elements of matrix $\boldsymbol{S}$, since (51) is equivalent to:

$$\begin{cases} \sum_{i+j=m, j>i} S_{ij}(j - i) = 0, \\ \sum_{i+j=m, j>i} S_{ij}(j^3 - i^3) = 0, \\ \sum_{i+j=m, j>i} S_{ij}(j^5 - i^5) = 0, \\ \sum_{i+j=m, j>i} S_{ij}(j^7 - i^7) = 0, \\ \sum_{i+j=m, j>i} S_{ij}(j^9 - i^9) = 0. \end{cases} \tag{52}$$

By checking rank of the coefficient matrix of any $m$, $\boldsymbol{S}$ must be zero matrix, then

**Theorem 2.6** $V_4^9$ can not be rotated horizontally.

# 3 The middle terms of harmonic sequence

In this section, we give another proof for the phenomenon that $V_4^9$ and $V_5^9$ can not be rotated horizontally. In fact, it is also right for any harmonic sequence in $\mathbb{C}P^9$.

**Theorem 3.1** In the following harmonic sequence associated to a linearly full holomorphic map $[f_0] = f: \boldsymbol{M}^2 \to \mathbb{C}P^9$,

$$0 \xrightarrow{\partial} \underline{f_0} \xrightarrow{\partial} \cdots \xrightarrow{\partial} \underline{f_4} \xrightarrow{\partial} \underline{f_5} \xrightarrow{\partial} \cdots \xrightarrow{\partial} \underline{f_9} \xrightarrow{\partial} 0. \tag{53}$$

$f_4, f_5$ can not be rotated horizontally.

**Proof** Assume $f_4$ can be rotated horizontally, ${}^t f_4 \boldsymbol{B} \mathrm{d} f_4 = 0$, from (11), (12), we have

$$\begin{aligned} {}^t f_4 \boldsymbol{B} \mathrm{d} f_4 &= {}^t f_4 \boldsymbol{B}(\partial f_4 \mathrm{d}z + \bar{\partial} f_4 \mathrm{d}\bar{z}) \\ &= {}^t f_4 \boldsymbol{B}(f_5 + \partial \log |f_4|^2 f_4) \mathrm{d}z + \\ &\quad {}^t f_4 \boldsymbol{B}(-\gamma_3 f_3) \mathrm{d}z \\ &= 0, \end{aligned} \tag{54}$$

Since $\boldsymbol{B}$ is antisymmetric, ${}^t f_4 \boldsymbol{B} f_4 = 0$, then it is equivalent to

$$\begin{cases} {}^t f_4 \boldsymbol{B} f_5 = 0, \\ {}^t f_4 \boldsymbol{B} f_3 = 0, \end{cases} \tag{55}$$

then we have

$${}^t f_4 \boldsymbol{B} f_5 = 0 \xrightarrow{\partial} {}^t f_4 \boldsymbol{B} f_6 = 0 \xrightarrow{\bar{\partial}} {}^t f_3 \boldsymbol{B} f_6 = 0. \tag{56}$$

Continue this process, we will get ${}^t f_0 \boldsymbol{B} f_9 = 0$, then do $\bar{\partial}$ continuously for it, we will get

$${}^t f_0 \boldsymbol{B} f_i = 0, i = 0,1,2,3,4,5,6,7,8,9, \tag{57}$$

then it is easy to check that $f_0$ must be a local zero section of the trivial bundle $\boldsymbol{M}^2 \times \mathbb{C}^{10}$, this is a contradiction.

$f_5$ is similar, we only need to show $f_9$ is a local zero section. □

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