

Stochastic Poisson integrators based on Padé approximations for linear stochastic Poisson systems^{*}

WANG Pengjun, WANG Lijin[†]
(School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing 100049, China)
(Received 4 April 2019; Revised 26 July 2019)

Wang P J, Wang L J. Stochastic Poisson integrators based on Padé approximations for linear stochastic Poisson systems [J]. Journal of University of Chinese Academy of Sciences, 2021,38(2):160-170.

Abstract We propose a class of stochastic Poisson integrators based on Padé approximations for linear stochastic Poisson systems. The root mean-square convergence orders of the schemes are analyzed, and the structure preserving properties are investigated. Numerical tests are performed to verify the theoretical results and illustrate the numerical behavior of the proposed methods.

Keywords stochastic Poisson integrators; root mean-square convergence order; Padé approximations; Poisson structure; Casimir functions

CLC number: O241.8 **Document code:** A **doi:**10.7523/j.issn.2095-6134.2021.02.002

线性随机泊松系统基于 Padé 近似的随机泊松积分子

王鹏钧, 王丽瑾
(中国科学院大学数学科学学院, 北京 100049)

摘要 利用 Padé 近似, 得到线性随机泊松系统的一类随机泊松积分子。证明数值格式的均方收敛阶, 及其对泊松结构和 Casimir 函数的保持。数值实验验证了数值格式的均方收敛阶及保结构特性。

关键词 随机泊松积分子; 均方收敛阶; Padé 近似; 泊松结构; Casimir 函数

Stochastic Poisson systems are defined as stochastic systems of the form^[1]

$$d\boldsymbol{X}(t) = \boldsymbol{B}(\boldsymbol{X})\left(\nabla H^0(\boldsymbol{X})dt + \sum_{i=1}^s \nabla H^i(\boldsymbol{X}) \circ dW^i(t)\right), \quad \boldsymbol{X}(t_0) = \boldsymbol{x}, \quad (1)$$

where $\boldsymbol{X}(t), \boldsymbol{x} \in \mathbb{R}^n, \boldsymbol{B}(\boldsymbol{X}) = (b_{ij}(\boldsymbol{X})) \in \mathbb{R}^{n \times n}$ is a skew-symmetric matrix-valued function satisfying

$$\sum_{l=1}^n \left(\frac{\partial b_{ij}(\boldsymbol{X})}{\partial X_l} b_{lk}(\boldsymbol{X}) + \frac{\partial b_{jk}(\boldsymbol{X})}{\partial X_l} b_{li}(\boldsymbol{X}) + \frac{\partial b_{ki}(\boldsymbol{X})}{\partial X_l} b_{lj}(\boldsymbol{X}) \right) = 0, \quad (2)$$

for all i, j, k . It is called the structural matrix. $H^i (i = 0, \dots, s)$ are smooth scalar functions, and $(W^1(t),$

^{*} Supported by the National Natural Science Foundation of China (11471310, 11071251)
[†] Corresponding author, E-mail: ljwang@ucas.ac.cn

$\cdots, W^s(t))$ is an s -dimensional standard Wiener process. The small circle “ \circ ” before dW^i denotes stochastic differential equations of Stratonovich sense.

If $n = 2d$ is even, and $\mathbf{B}(\mathbf{X}) \equiv \mathbf{J} = \begin{pmatrix} 0 & -\mathbf{I}_d \\ \mathbf{I}_d & 0 \end{pmatrix}$, where \mathbf{I}_d is the d -dimensional identity matrix, the stochastic Poisson systems (1) degenerate to the stochastic Hamiltonian systems^[2-4] of even dimensions $2d$. If $H^i \equiv 0$ for $i = 1, \cdots, s$, the stochastic Poisson systems (1) degenerate to the deterministic Poisson systems^[5-7], whose long history goes back to the 19th century. Their stochastic counterparts, i. e., the stochastic Poisson systems (1), however, got attention, to our knowledge, only in recent years (See Refs. [1, 8-9]). As was pointed out in Refs. [5-7], etc., Poisson systems are generalizations of Hamiltonian systems on Poisson manifolds, and find applications in a vast variety of fields such as rigid bodies, quantum mechanics, satellite orbits, magnetization fluid dynamics, etc.

It can be proved that^[1], almost surely, the phase flow $\phi_t: \mathbf{x} \mapsto \mathbf{X}(t)$ of the stochastic Poisson systems (1) preserves the Poisson structure

$$\frac{\partial \mathbf{X}(t)}{\partial \mathbf{x}} \mathbf{B}(\mathbf{x}) \frac{\partial \mathbf{X}(t)}{\partial \mathbf{x}}^T = \mathbf{B}(\mathbf{X}(t)), \quad \forall t \geq 0. \quad (3)$$

As was given in Refs. [1, 5-7], etc., a function $\mathbf{C}(\mathbf{X})$ is called a Casimir function of a Poisson system with structural matrix $\mathbf{B}(\mathbf{X})$, if

$$\nabla \mathbf{C}(\mathbf{X})^T \mathbf{B}(\mathbf{X}) \equiv 0, \text{ for all } \mathbf{X}. \quad (4)$$

It is not difficult to prove that (see Ref. [1]), the Casimir functions $\mathbf{C}(\mathbf{X})$ are invariants of the stochastic Poisson systems (1).

Numerical methods preserve the Poisson structure and the Casimir functions, namely, the methods $\{\mathbf{X}_n\}$ satisfying

$$\begin{aligned} \frac{\partial \mathbf{X}_{n+1}}{\partial \mathbf{X}_n} \mathbf{B}(\mathbf{X}_n) \frac{\partial \mathbf{X}_{n+1}}{\partial \mathbf{X}_n}^T &= \mathbf{B}(\mathbf{X}_{n+1}), \\ \mathbf{C}(\mathbf{X}_{n+1}) &= \mathbf{C}(\mathbf{X}_n), \quad \forall n \geq 0, \end{aligned} \quad (5)$$

are called Poisson integrators^[6]. Poisson integrators for deterministic Poisson systems have been developed from various points of view (see Refs.

[5-7] and references therein). Numerical methods for stochastic Poisson systems, however, are still rarely studied, except in a number of recent papers on structure-preserving methods for certain special stochastic Poisson systems with single noise or/and even dimensions (e. g., Refs. [8-9]), as well as our paper (Ref. [1]) on numerical methods based on Darboux-Lie theorem for general stochastic Poisson systems.

In this paper, we consider linear stochastic Poisson systems of the form

$$\begin{aligned} d\mathbf{X}(t) &= \mathbf{B}(\nabla H^0(\mathbf{X}) dt + \sum_{i=1}^s \nabla H^i(\mathbf{X}) \circ dW^i(t)), \\ \mathbf{X}(t_0) &= \mathbf{x}, \end{aligned} \quad (6)$$

where $\mathbf{B} \in \mathbb{R}^{n \times n}$ is a constant skew-symmetric matrix, and $H^i(\mathbf{X}) = \frac{1}{2} \mathbf{X}^T \mathbf{C}^i \mathbf{X}$, where $\mathbf{C}^i, i = 0, \cdots, n$, are constant symmetric matrices. Then (6) can be rewritten as

$$\begin{aligned} d\mathbf{X}(t) &= \mathbf{A}^0 \mathbf{X} dt + \sum_{i=1}^s \mathbf{A}^i \mathbf{X} \circ dW^i(t), \\ \mathbf{X}(t_0) &= \mathbf{x}, \end{aligned} \quad (7)$$

where

$$\mathbf{A}^i = \mathbf{B} \mathbf{C}^i, i = 0, \cdots, s.$$

The exact solution of (7) is of the form $\mathbf{X}(t) =$

$$\exp \left[(t - t_0) \mathbf{A}^0 + \sum_{i=1}^s (W^i(t) - W^i(t_0)) \mathbf{A}^i \right] \cdot \mathbf{X}(t_0). \quad (8)$$

On the analogy of the discussions for linear Hamiltonian systems in Refs. [5, 10-11], we need to simulate the random matrix exponential in (8) appropriately to obtain high order structure-preserving numerical solvers.

As mentioned above, as $\mathbf{B} \equiv \mathbf{J} = \begin{pmatrix} 0 & -\mathbf{I}_d \\ \mathbf{I}_d & 0 \end{pmatrix}$

with $n = 2d$, the linear stochastic Poisson systems (7) degenerate to the linear stochastic Hamiltonian systems (SHSs). If, further, $\mathbf{A}^i = 0$ for $i = 1, \cdots, s$, (7) will degenerate to the linear deterministic Hamiltonian systems (DHSs). In Ref. [10], Feng et al. simulated the matrix exponential $\exp[(t_{n+1} -$

$t_n)A^0]$ appearing in the exact solution of the linear DHSs, by appropriate Padé approximations, and obtained high-order symplectic schemes. This approach was generalized in Ref. [11] to linear SHSs, where random matrix exponentials $\exp[(t_{n+1} - t_n)A^0 + \sum_{i=1}^s (W^i(t_{n+1}) - W^i(t_n))A^i]$ with $A^i = JC^i$ ($i = 0, \dots, s$) were approximated by suitable Padé approximations, to create stochastic symplectic methods of high root mean-square orders.

In this paper, we apply Padé approximations to the random matrix exponential in (8), to produce stochastic Poisson integrators which preserve both the Poisson structure and the Casimir functions of the original linear stochastic Poisson systems (7) and possess high root mean-square accuracy. This is an extension of the above-mentioned Padé approximation approaches for linear Hamiltonian systems^[5,10-11] to linear stochastic Poisson context.

The contents of this paper are organized as follows. In section 1, we construct the numerical schemes based on the Padé approximations, namely, the (l, m) -schemes. In section 2 we analyze the root mean-square convergence orders of the proposed (l, m) -schemes. In section 3 we prove the preservation of the Poisson structure and the Casimir functions by the (p, p) -schemes, that is, the (l, m) -schemes when $l = m = p$. Numerical experiments are performed in section 4 to verify the theoretical analysis and illustrate the numerical behavior of the proposed schemes. Section 5 is a brief conclusion.

1 Numerical schemes based on the Padé approximations

To simulate the random matrix exponential in (8), we can use the following Padé approximation (see e. g. Refs. [10,12])

$\exp(\boldsymbol{M}) \approx \boldsymbol{P}_{l,m}(\boldsymbol{M}) = \boldsymbol{D}_{l,m}(\boldsymbol{M})\boldsymbol{N}_{l,m}(\boldsymbol{M})$, (9) where \boldsymbol{M} represents an $n \times n$ dimensional square matrix, l, m are positive integers, and

$$\boldsymbol{D}_{l,m}(\boldsymbol{M}) = \boldsymbol{I} + \sum_{k=1}^l d_k^{l,m}(-\boldsymbol{M})^k, \\ \boldsymbol{N}_{l,m}(\boldsymbol{M}) = \boldsymbol{I} + \sum_{k=1}^m n_k^{l,m}\boldsymbol{M}^k$$

with

$$d_k^{l,m} = \frac{(l+m-k)! l!}{(l+m)! k! (l-k)!}, \\ n_k^{l,m} = \frac{(l+m-k)! m!}{(l+m)! k! (m-k)!}.$$

The invertibility of the matrix $\boldsymbol{D}_{l,m}(\boldsymbol{M})$ can be ensured by e. g. $|\sum_{k=1}^l d_k^{l,m}(-\boldsymbol{M})^k| < 1$. According to Ref. [5], when $|\boldsymbol{M}|$ is small and $|\boldsymbol{M}| \rightarrow 0$, we have

$$|\exp(\boldsymbol{M}) - \boldsymbol{P}_{l,m}(\boldsymbol{M})| = O(|\boldsymbol{M}|^{l+m+1}). \tag{10}$$

Now we construct the following numerical discretization for the linear stochastic Poisson system (7)

$$\boldsymbol{X}_{n+1}^{l,m} = [\boldsymbol{I} + \sum_{k=1}^l d_k^{l,m}(-\boldsymbol{Z}_n)^k]^{-1} \cdot \\ [\boldsymbol{I} + \sum_{k=1}^m n_k^{l,m}(\boldsymbol{Z}_n)^k] \boldsymbol{X}_n^{l,m}, \\ \boldsymbol{X}_0^{l,m} = \boldsymbol{x}, \tag{11}$$

where

$$\boldsymbol{Z}_n = h\boldsymbol{A}^0 + \sum_{i=1}^s [W^i(t_{n+1}) - W^i(t_n)]\boldsymbol{A}^i, \\ t_n = t_0 + nh.$$

$W^i(t_{n+1}) - W^i(t_n)$ can be simulated by $\sqrt{h}\xi_n^i$, and ξ_n^i ($i = 1, \dots, s$) are $N(0,1)$ -distributed random variables. To guarantee the invertibility of the matrix $\boldsymbol{I} + \sum_{k=1}^l d_k^{l,m}(-\boldsymbol{Z}_n)^k$ in the scheme (11), we can require $|\sum_{k=1}^l d_k^{l,m}(-\boldsymbol{Z}_n)^k| < 1$, which can be realized by choosing sufficiently small h , and truncating the random variables ξ_n^i as follows^[3]

$$\xi_{n,h}^i = \begin{cases} \xi_n^i, & |\xi_n^i| \leq A_h, \\ A_h, & \xi_n^i > A_h, \\ -A_h, & \xi_n^i < -A_h, \end{cases} \tag{12}$$

$A_h = \sqrt{2c|\ln h|}$, $c \geq 1$ sufficiently large. The truncation was proposed in Ref. [3] for constructing implicit stochastic methods, where the unboundedness of $\Delta W_n^i = W^i(t_{n+1}) - W^i(t_n)$ may cause zero denominators and thus collapse of the methods. It was shown in Ref. [3] that, the root mean-square convergence order ν of the proposed methods will not be affected by such truncations if c

is sufficiently large such that $c \geq 2\nu$.

After the truncation (12), we rewrite the scheme (11) in the following form

$$\begin{aligned} \mathbf{X}_{n+1}^{l,m} &= \left[-\sum_{k=1}^l d_k^{l,m} (-\overline{\mathbf{Z}}_n)^k \right] \mathbf{X}_{n+1}^{l,m} + \\ &\quad \left[\mathbf{I} + \sum_{k=1}^m n_k^{l,m} \overline{\mathbf{Z}}_n^k \right] \mathbf{X}_n^{l,m}, \\ \mathbf{X}_0^{l,m} &= \mathbf{x}, \end{aligned} \tag{13}$$

where

$$\overline{\mathbf{Z}}_n = h\mathbf{A}^0 + \sum_{i=1}^s \sqrt{h} \zeta_{n,h}^i \mathbf{A}^i. \tag{14}$$

We can use fixed-point iterations to get $\mathbf{X}_{n+1}^{l,m}$ in (13), where the condition $|\sum_{k=1}^l d_k^{l,m} (-\overline{\mathbf{Z}}_n)^k| < 1$ guaranteed by the truncation (12) as well as sufficiently small h satisfying

$$\sqrt{h} |\ln h| < \min \left\{ \frac{1}{\sqrt{2c}}, \frac{1}{\sqrt{2c} \cdot \max_k \{d_k^{l,m}\} \cdot Q_l} \right\}, \tag{15}$$

with $Q_l = \sum_{k=1}^l (\sum_{i=0}^s |A^i|)^k$, can as well ensure the convergence of the iterations. In the following we call the scheme (13) the (l,m) -scheme.

2 Root mean-square convergence orders

Proposition 2.1^[13] Suppose the one-step approximation $\overline{\mathbf{X}}_{t,x}(t+h)$ has order of accuracy p_1 for the mathematical expectation of the deviation and order of accuracy p_2 for the mean-square deviation; more precisely, for arbitrary $t_0 \leq t \leq t_0 + T - h, \mathbf{x} \in \mathbb{R}^n$, the following inequalities hold:

$$\begin{aligned} |E(\mathbf{X}_{t,x}(t+h) - \overline{\mathbf{X}}_{t,x}(t+h))| &\leq K(1 + |\mathbf{x}|^2)^{1/2} h^{p_1}, \\ [E|\mathbf{X}_{t,x}(t+h) - \overline{\mathbf{X}}_{t,x}(t+h)|^2]^{1/2} &\leq K(1 + |\mathbf{x}|^2)^{1/2} h^{p_2}. \end{aligned} \tag{16}$$

Also, let

$$p_2 \geq \frac{1}{2}, p_1 \geq p_2 + \frac{1}{2}. \tag{17}$$

Then for any N and $k = 0, 1, \dots, N$ the following inequality holds:

$$\begin{aligned} [E|\mathbf{X}_{t_0, \mathbf{x}_0}(t_k) - \overline{\mathbf{X}}_{t_0, \mathbf{x}_0}(t_k)|^2]^{1/2} &\leq K(1 + E|\mathbf{X}_0|^2)^{1/2} h^{p_2 - 1/2}, \end{aligned} \tag{18}$$

i. e. the order of accuracy of the method constructed using the one-step approximation $\overline{\mathbf{X}}_{t,x}(t+h)$ is $p = p_2 - \frac{1}{2}$.

Proposition 2.2^[13] Let the one-step approximation $\overline{\mathbf{X}}_{t,x}(t+h)$ satisfy the conditions of Theorem 2.1. Suppose that $\widetilde{\mathbf{X}}_{t,x}(t+h)$ is such that

$$\begin{aligned} |E(\overline{\mathbf{X}}_{t,x}(t+h) - \widetilde{\mathbf{X}}_{t,x}(t+h))| &= O(h^{p_1}), \\ [E|\overline{\mathbf{X}}_{t,x}(t+h) - \widetilde{\mathbf{X}}_{t,x}(t+h)|^2]^{1/2} &= O(h^{p_2}) \end{aligned} \tag{19}$$

with the same h^{p_1} and h^{p_2} . Then the method based on the one-step approximation $\widetilde{\mathbf{X}}_{t,x}(t+h)$ has the same root mean-square convergence order as the method based on $\overline{\mathbf{X}}_{t,x}(t+h)$, i. e., its root-mean-square convergence order is $p = p_2 - \frac{1}{2}$.

Using the above fundamental convergence theorem and its corollary, namely the Propositions 2.1 and 2.2, we can prove the following convergence theorem for the scheme (13).

Theorem 2.1 If l and m have the same parity, and the truncation parameter c in (12) satisfies $c \geq l + m$, then the root mean-square convergence order of the (l,m) -scheme (13) is $\frac{l+m}{2}$.

Proof Consider the following scheme:

$$\begin{aligned} \overline{\mathbf{X}}_{n+1}^{l,m} &= \overline{\mathbf{X}}_n^{l,m} + \left[\sum_{j=1}^{l+m} \frac{1}{j!} \overline{\mathbf{Z}}_n^j \right] \overline{\mathbf{X}}_n^{l,m}, \\ \overline{\mathbf{X}}_0^{l,m} &= \mathbf{x}, \end{aligned} \tag{20}$$

which is based on the one-step approximation

$$\begin{aligned} \overline{\mathbf{X}}^{l,m}(t+h; t, \mathbf{x}) &= \mathbf{x} + \\ &\quad \sum_{j=1}^{l+m} \frac{1}{j!} (h\mathbf{A}^0 + \sum_{i=1}^s \sqrt{h} \zeta_h^i \mathbf{A}^i)^j \mathbf{x}, \end{aligned} \tag{21}$$

where ζ_h^i is the truncation of the $N(0,1)$ random variable ξ^i that makes $W^i(t+h) - W^i(t) = \sqrt{h} \xi^i$. Using Taylor expansion of the exponential, we have the exact solution

$$\mathbf{X}(t+h; t, \mathbf{x}) = \mathbf{x} + \sum_{j=1}^{+\infty} \frac{1}{j!} (h\mathbf{A}^0 + \sum_{i=1}^s \sqrt{h} \xi^i \mathbf{A}^i)^j \mathbf{x}. \tag{22}$$

Then we can derive

$$\begin{aligned} & |E(X(t+h; t, \mathbf{x}) - \overline{X^{l,m}}(t+h; t, \mathbf{x}))| \\ & \leq |E[\sum_{j=1}^{l+m} \frac{1}{j!} (h\mathbf{A}^0 + \sum_{i=1}^s \sqrt{h}\xi^i \mathbf{A}^i)^j - \\ & \quad \sum_{j=1}^{l+m} \frac{1}{j!} (h\mathbf{A}^0 + \sum_{i=1}^s \sqrt{h}\xi_h^i \mathbf{A}^i)^j + \\ & \quad \sum_{j=l+m+1}^{+\infty} \frac{1}{j!} (h\mathbf{A}^0 + \sum_{i=1}^s \sqrt{h}\xi^i \mathbf{A}^i)^j]| \cdot |\mathbf{x}| \\ & \leq (1 + |\mathbf{x}|^2)^{\frac{1}{2}} (L_1 + L_2 + L_3), \end{aligned} \quad (23)$$

where

$$\begin{aligned} L_1 &= \sum_{j=1}^{l+m} \frac{1}{j!} \\ & |E[(h\mathbf{A}^0 + \sum_{i=1}^s \sqrt{h}\xi^i \mathbf{A}^i)^j - (h\mathbf{A}^0 + \sum_{i=1}^s \sqrt{h}\xi_h^i \mathbf{A}^i)^j]|, \\ L_2 &= \frac{1}{(l+m+1)!} |E(h\mathbf{A}^0 + \sum_{i=1}^s \sqrt{h}\xi^i \mathbf{A}^i)^{(l+m+1)}|, \\ L_3 &= \sum_{j=l+m+2}^{+\infty} \frac{1}{j!} |E(h\mathbf{A}^0 + \sum_{i=1}^s \sqrt{h}\xi^i \mathbf{A}^i)^j|. \end{aligned}$$

Due to the fact that, for square matrices $\mathbf{M}^i (i = 0, \dots, s)$, and any $j \geq 1 (j \in \mathbb{Z})$,

$$\left(\sum_{i=0}^s \mathbf{M}^i\right)^j = \sum_{\sigma_1, \dots, \sigma_j \in \{0, 1, \dots, s\}} \mathbf{M}^{\sigma_1} \dots \mathbf{M}^{\sigma_j},$$

and $h < h^{\frac{1}{2}}$ for $0 < h < 1$, by denoting $\xi^0 = 1, \xi_h^0 = 1$, we have

$$\begin{aligned} L_1 &\leq \sum_{j=1}^{l+m} \frac{1}{j!} h^{\frac{j}{2}} |E \sum_{\sigma_1, \dots, \sigma_j \in \{0, \dots, s\}} \mathbf{A}^{\sigma_1} \dots \mathbf{A}^{\sigma_j} (\xi^{\sigma_1} \dots \xi^{\sigma_j} - \xi_h^{\sigma_1} \dots \xi_h^{\sigma_j})| \\ &\leq \sum_{j=1}^{l+m} \frac{1}{j!} h^{\frac{j}{2}} \max_{0 \leq i \leq s} |\mathbf{A}^i|^j \sum_{\sigma_1, \dots, \sigma_j \in \{0, \dots, s\}} |E(\xi^{\sigma_1} \dots \xi^{\sigma_j} - \\ &\quad \xi_h^{\sigma_1} \dots \xi_h^{\sigma_j})| \\ &= \sum_{j=1}^{l+m} \frac{1}{j!} h^{\frac{j}{2}} \max_{0 \leq i \leq s} |\mathbf{A}^i|^j \sum_{0 < \rho_1 + \dots + \rho_s \leq j} \\ &\quad |E[(\xi^1)^{\rho_1} \dots (\xi^s)^{\rho_s} - (\xi_h^1)^{\rho_1} \dots (\xi_h^s)^{\rho_s}]|, \end{aligned}$$

where $\rho_i \geq 0, \rho_i \in \mathbb{Z} (i = 1, \dots, s)$.

Obviously, $L_2 \leq O(h^{\frac{l+m+1}{2}})$, where the principal term, i. e. the term with lowest order, denoted by P_2 , satisfies

$$\begin{aligned} P_2 &\leq \frac{K_0}{(l+m+1)!} h^{\frac{l+m+1}{2}} \\ & |E \sum_{\sigma_1, \dots, \sigma_{l+m+1} \in \{1, \dots, s\}} \mathbf{A}^{\sigma_1} \dots \mathbf{A}^{\sigma_{l+m+1}} \xi^{\sigma_1} \dots \xi^{\sigma_{l+m+1}}| \\ &\leq \frac{K_0}{(l+m+1)!} h^{\frac{l+m+1}{2}} \max_{1 \leq i \leq s} |\mathbf{A}^i|^{l+m+1} \end{aligned}$$

$$\begin{aligned} & \sum_{\sigma_1, \dots, \sigma_{l+m+1} \in \{1, \dots, s\}} |E(\xi^{\sigma_1} \dots \xi^{\sigma_{l+m+1}})| \\ &= \frac{K_0}{(l+m+1)!} h^{\frac{l+m+1}{2}} \max_{1 \leq i \leq s} |\mathbf{A}^i|^{l+m+1} \\ & \sum_{\rho_1 + \dots + \rho_s = l+m+1} |E[(\xi^1)^{\rho_1} \dots (\xi^s)^{\rho_s}]|, \end{aligned}$$

where K_0 is a positive constant independent of h . Then, if $l+m+1$ is odd, due to independence of ξ^1, \dots, ξ^s , we have $P_2 = 0$ which implies $L_2 \leq O(h^{\frac{l+m+2}{2}})$. As $l+m+1$ is even, $L_2 \leq O(h^{\frac{l+m+1}{2}})$.

According to the distributions of ξ^i and ξ_h^i , and let $t = x - A_h$, we have for any $\rho \geq 1 (\rho \in \mathbb{Z})$,

$$\begin{aligned} & |E[(\xi^i)^\rho - (\xi_h^i)^\rho]| \\ &= 2 \left| \int_{A_h}^{+\infty} (x^\rho - A_h^\rho) \exp\left(-\frac{x^2}{2}\right) dx \right| \\ &\leq 2 \left| \sum_{u=0}^{\rho-1} C_\rho^u A_h^u \exp\left(-\frac{A_h^2}{2}\right) \int_0^{+\infty} t^{\rho-u} \exp\left(-\frac{t^2}{2}\right) dt \right| \\ &\leq 2K_1 \left| \sum_{u=0}^{\rho-1} A_h^u \exp\left(-\frac{A_h^2}{2}\right) \right|, \end{aligned} \quad (24)$$

where

$$K_1 = \max_{0 \leq u \leq \rho-1} C_\rho^u \rho! \sqrt{\pi/2},$$

and the last inequality is due to

$$\begin{aligned} & \int_0^{+\infty} t^\nu \exp\left(-\frac{t^2}{2}\right) dt \\ &= \begin{cases} (\nu-1)(\nu-3)\dots 4 \times 2, & \nu \text{ is odd,} \\ (\nu-1)(\nu-3)\dots 5 \times 3 \sqrt{\pi/2}, & \nu \text{ is even,} \end{cases} \end{aligned}$$

for $\nu = \rho - u$ which satisfies $1 \leq \nu \leq \rho$.

Since $A_h = \sqrt{2c \lfloor \ln h \rfloor}$, $c \geq l+m$, we have $\exp(-\frac{A_h^2}{2}) \leq h^c$, $A_h^2 \leq \frac{2c}{h}$, which implies

$$|E[(\xi^i)^\rho - (\xi_h^i)^\rho]| \leq 2K_1 K_2 h^{\frac{1-\rho}{2} + l+m} \quad (25)$$

for $i = 1, \dots, s$. Note that K_1, K_2 are positive constants independent of h .

On the other hand,

$$\begin{aligned} & |E[(\xi^1)^{\rho_1} (\xi^2)^{\rho_2} \dots (\xi^{s-1})^{\rho_{s-1}} (\xi^s)^{\rho_s} - \\ & \quad (\xi_h^1)^{\rho_1} (\xi_h^2)^{\rho_2} \dots (\xi_h^{s-1})^{\rho_{s-1}} (\xi_h^s)^{\rho_s}]| \\ &\leq |E[(\xi^1)^{\rho_1} (\xi^2)^{\rho_2} \dots (\xi^{s-1})^{\rho_{s-1}} (\xi^s)^{\rho_s} - \\ & \quad (\xi_h^1)^{\rho_1} (\xi^2)^{\rho_2} \dots (\xi^{s-1})^{\rho_{s-1}} (\xi^s)^{\rho_s}]| + \\ & |E[(\xi_h^1)^{\rho_1} (\xi^2)^{\rho_2} \dots (\xi^{s-1})^{\rho_{s-1}} (\xi^s)^{\rho_s} - \\ & \quad (\xi_h^1)^{\rho_1} (\xi_h^2)^{\rho_2} \dots (\xi^{s-1})^{\rho_{s-1}} (\xi^s)^{\rho_s}]| + \\ & \dots + \end{aligned}$$

$$\begin{aligned}
& | E [(\zeta_h^1)^{\rho_1} (\zeta_h^2)^{\rho_2} \dots (\zeta_h^{s-1})^{\rho_{s-1}} (\xi^s)^{\rho_s} - \\
& (\zeta_h^1)^{\rho_1} (\zeta_h^2)^{\rho_2} \dots (\zeta_h^{s-1})^{\rho_{s-1}} (\zeta_h^s)^{\rho_s}] | \\
= & | E [(\xi^1)^{\rho_1} - (\zeta_h^1)^{\rho_1}] | + \\
& | E [(\xi^2)^{\rho_2} \dots (\xi^{s-1})^{\rho_{s-1}} (\xi^s)^{\rho_s}] | + \\
& | E [(\xi^2)^{\rho_2} - (\zeta_h^2)^{\rho_2}] | + \\
& | E [(\zeta_h^1)^{\rho_1} \dots (\xi^{s-1})^{\rho_{s-1}} (\xi^s)^{\rho_s}] | + \\
& \dots + \\
& | E [(\xi^s)^{\rho_s} - (\zeta_h^s)^{\rho_s}] | + \\
& | E [(\zeta_h^1)^{\rho_1} \dots (\zeta_h^{s-1})^{\rho_{s-1}}] |. \quad (26)
\end{aligned}$$

From (25) we know that

$$| E [(\xi^i)^\rho - (\zeta_h^i)^\rho] | = O(h^{\frac{1-\rho}{2} + l+m}),$$

which, together with (26), implies that

$$L_1 = O(h^{\frac{l+m+1}{2}}). \quad (27)$$

We have analyzed the order of L_2 above. In addition,

$$L_3 \leq O(h^{\frac{l+m+2}{2}}) \text{ obviously. Then we obtain}$$

$$\begin{aligned}
& | E (X(t+h; t, \mathbf{x}) - \overline{X^{l,m}}(t+h; t, \mathbf{x})) | \\
& \leq K(1 + |\mathbf{x}|^2)^{1/2} h^{\lceil \frac{l+m+1}{2} \rceil}, \quad (28)
\end{aligned}$$

where K is a positive constant independent of h , and

$\lceil \frac{l+m+1}{2} \rceil$ means the smallest integer larger than

or equal to $\frac{l+m+1}{2}$. Therefore, when l and m have

the same parity, namely $l+m+1$ is odd, the order

of (28) would be $\frac{l+m}{2} + 1$. Otherwise, it would be

$$\frac{l+m+1}{2}.$$

Now we estimate the second moment of the error. Applying the inequality

$$E | \sum_{i=1}^k a_i |^2 \leq k (\sum_{i=1}^k E | a_i |^2) \quad (29)$$

we have

$$\begin{aligned}
& E | X(t+h; t, \mathbf{x}) - \overline{X^{l,m}}(t+h; t, \mathbf{x}) |^2 \\
& \leq E | \sum_{j=1}^{l+m} \frac{1}{j!} (hA^0 + \sum_{i=1}^s \sqrt{h} \xi^i A^i)^j - \\
& \sum_{j=1}^{l+m} \frac{1}{j!} (hA^0 + \sum_{i=1}^s \sqrt{h} \zeta_h^i A^i)^j + \\
& \sum_{j=l+m+1}^{+\infty} \frac{1}{j!} (hA^0 + \sum_{i=1}^s \sqrt{h} \xi^i A^i)^j |^2 \cdot |\mathbf{x}|^2 \\
& \leq 3 |\mathbf{x}|^2 (L_4 + L_5 + L_6), \quad (30)
\end{aligned}$$

where

$$L_4 = E \left| \sum_{j=1}^{l+m} \frac{1}{j!} [(hA^0 + \sum_{i=1}^s \sqrt{h} \xi^i A^i)^j - (hA^0 + \sum_{i=1}^s \sqrt{h} \zeta_h^i A^i)^j] \right|^2,$$

$$L_5 = E \left| \frac{1}{(l+m+1)!} (hA^0 + \sum_{i=1}^s \sqrt{h} \xi^i A^i)^{(l+m+1)} \right|^2,$$

$$L_6 = E \left| \sum_{j=l+m+2}^{+\infty} \frac{1}{j!} (hA^0 + \sum_{i=1}^s \sqrt{h} \xi^i A^i)^j \right|^2.$$

Again setting $\xi^0 = 1, \zeta_h^0 = 1$, and applying (29) we have

$$\begin{aligned}
L_4 & \leq (l+m) \sum_{j=1}^{l+m} \frac{1}{(j!)^2} \\
& E \left| (hA^0 + \sum_{i=1}^s \sqrt{h} \xi^i A^i)^j - (hA^0 + \sum_{i=1}^s \sqrt{h} \zeta_h^i A^i)^j \right|^2 \\
& \leq (l+m) \sum_{j=1}^{l+m} \frac{1}{(j!)^2} h^j E \left| \sum_{\sigma_1, \dots, \sigma_j \in \{0, \dots, s\}} A^{\sigma_1} \dots A^{\sigma_j} (\xi^{\sigma_1} \dots \xi^{\sigma_j} - \zeta_h^{\sigma_1} \dots \zeta_h^{\sigma_j}) \right|^2 \\
& \leq (l+m) \sum_{j=1}^{l+m} \frac{(s+1)^j}{(j!)^2} \max_{0 \leq i \leq s} |A^i|^{2j} h^j \\
& \sum_{\sigma_1, \dots, \sigma_j \in \{0, \dots, s\}} E | \xi^{\sigma_1} \dots \xi^{\sigma_j} - \zeta_h^{\sigma_1} \dots \zeta_h^{\sigma_j} |^2 \\
& = (l+m) \sum_{j=1}^{l+m} \frac{(s+1)^j}{(j!)^2} \max_{0 \leq i \leq s} |A^i|^{2j} h^j \sum_{0 < \rho_1 + \dots + \rho_s \leq j} \\
& E | (\xi^1)^{\rho_1} \dots (\xi^s)^{\rho_s} - (\zeta_h^1)^{\rho_1} \dots (\zeta_h^s)^{\rho_s} |^2.
\end{aligned}$$

Meanwhile, it is easy to see that $L_5 \leq O(h^{l+m+1})$,

the principal term of which, denoted by P_5 , satisfies

$$\begin{aligned}
P_5 & \leq K_3 h^{l+m+1} E \left| \sum_{\sigma_1, \dots, \sigma_{l+m+1} \in \{1, \dots, s\}} A^{\sigma_1} \dots A^{\sigma_{l+m+1}} \xi^{\sigma_1} \dots \xi^{\sigma_{l+m+1}} \right|^2 \\
& \leq K_3 h^{l+m+1} s^{l+m+1} \max_{1 \leq i \leq s} |A^i|^{2(l+m+1)} \\
& \sum_{\sigma_1, \dots, \sigma_{l+m+1} \in \{1, \dots, s\}} E | \xi^{\sigma_1} \dots \xi^{\sigma_{l+m+1}} |^2 \\
& = K_3 h^{l+m+1} s^{l+m+1} \max_{1 \leq i \leq s} |A^i|^{2(l+m+1)} \\
& \sum_{\rho_1 + \dots + \rho_s = l+m+1} E | (\xi^1)^{\rho_1} \dots (\xi^s)^{\rho_s} |^2 \\
& = O(h^{l+m+1}),
\end{aligned}$$

where K_3 is a positive constant independent of h . Thus

$L_5 \leq O(h^{l+m+1})$, and obviously, $L_6 \leq O(h^{l+m+2})$.

Similar to the analysis for (25), let $t = x - A_h$, we have for any $\rho \geq 1$ ($\rho \in \mathbb{Z}$) that

$$\begin{aligned}
& E | (\xi^i)^\rho - (\xi_h^i)^\rho |^2 \\
&= 2 \int_{A_h}^{+\infty} (x^\rho - A_h^\rho)^2 \exp\left(-\frac{x^2}{2}\right) dx \\
&= 2 \int_0^{+\infty} \left(\sum_{u=0}^{\rho-1} C_\rho^u A_h^u t^{\rho-u}\right)^2 \exp\left(-\frac{(t+A_h)^2}{2}\right) dt \\
&\leq 2 \int_0^{+\infty} \rho \left(\sum_{u=0}^{\rho-1} (C_\rho^u)^2 A_h^{2u} t^{2\rho-2u}\right) \exp\left(-\frac{(t+A_h)^2}{2}\right) dt \\
&\leq 2\rho \sum_{u=0}^{\rho-1} (C_\rho^u)^2 A_h^{2u} \exp\left(-\frac{A_h^2}{2}\right) \int_0^{+\infty} t^{2\rho-2u} \exp\left(-\frac{t^2}{2}\right) dt \\
&\leq 2\rho K_4 \sum_{u=0}^{\rho-1} (C_\rho^u)^2 A_h^{2u} \exp\left(-\frac{A_h^2}{2}\right) \\
&\leq 2\rho K_5 h^{1-\rho+l+m}, \tag{31}
\end{aligned}$$

for $i = 1, \dots, s$, where K_4, K_5 are positive constants independent of h .

For L_4 , we have

$$\begin{aligned}
& \frac{1}{s} E | (\xi^1)^{\rho_1} (\xi^2)^{\rho_2} \dots (\xi^{s-1})^{\rho_{s-1}} (\xi^s)^{\rho_s} - \\
& (\xi_h^1)^{\rho_1} (\xi_h^2)^{\rho_2} \dots (\xi_h^{s-1})^{\rho_{s-1}} (\xi_h^s)^{\rho_s} |^2 \\
&\leq E | (\xi^1)^{\rho_1} (\xi^2)^{\rho_2} \dots (\xi^{s-1})^{\rho_{s-1}} (\xi^s)^{\rho_s} - \\
& (\xi_h^1)^{\rho_1} (\xi_h^2)^{\rho_2} \dots (\xi_h^{s-1})^{\rho_{s-1}} (\xi_h^s)^{\rho_s} |^2 + \\
& E | (\xi_h^1)^{\rho_1} (\xi_h^2)^{\rho_2} \dots (\xi_h^{s-1})^{\rho_{s-1}} (\xi_h^s)^{\rho_s} - \\
& (\xi_h^1)^{\rho_1} (\xi_h^2)^{\rho_2} \dots (\xi_h^{s-1})^{\rho_{s-1}} (\xi_h^s)^{\rho_s} |^2 + \\
& \dots + \\
& E | (\xi_h^1)^{\rho_1} (\xi_h^2)^{\rho_2} \dots (\xi_h^{s-1})^{\rho_{s-1}} (\xi_h^s)^{\rho_s} - \\
& (\xi_h^1)^{\rho_1} (\xi_h^2)^{\rho_2} \dots (\xi_h^{s-1})^{\rho_{s-1}} (\xi_h^s)^{\rho_s} |^2 \\
&= E | (\xi^1)^{\rho_1} - (\xi_h^1)^{\rho_1} |^2 \cdot \\
& E | (\xi^2)^{\rho_2} \dots (\xi^{s-1})^{\rho_{s-1}} (\xi^s)^{\rho_s} |^2 + \\
& E | (\xi^2)^{\rho_2} - (\xi_h^2)^{\rho_2} |^2 \cdot \\
& E | (\xi_h^1)^{\rho_1} \dots (\xi^{s-1})^{\rho_{s-1}} (\xi^s)^{\rho_s} |^2 + \\
& \dots + \\
& E | (\xi^s)^{\rho_s} - (\xi_h^s)^{\rho_s} |^2 \cdot \\
& E | (\xi_h^1)^{\rho_1} \dots (\xi_h^{s-1})^{\rho_{s-1}} |^2. \tag{32}
\end{aligned}$$

From (31) we know that for $i \in \{1, \dots, s\}$,

$$E | (\xi^i)^\rho - (\xi_h^i)^\rho |^2 = O(h^{1-\rho+l+m}),$$

which, together with (32), implies that

$$L_4 \leq O(h^{l+m+1}). \tag{33}$$

Considering also the orders of L_5 and L_6 discussed above, we obtain

$$[E | \mathbf{X}(t+h; t, \mathbf{x}) - \overline{\mathbf{X}^{l,m}}(t+h; t, \mathbf{x}) |^2]^{1/2}$$

$$\leq L(1 + |\mathbf{x}|^2)^{1/2} h^{\frac{l+m+1}{2}}, \tag{34}$$

where L is a positive constant independent of h .

Applying Proposition 2.1, for the case that l

and m have the same parity, we have $p_1 = \frac{l+m}{2} + 1$,

$$p_2 = \frac{l+m+1}{2}. \quad \text{So the root mean-square convergence order of the numerical scheme (20) is } \frac{l+m}{2}.$$

Next we consider the difference:

$$\begin{aligned}
& \mathbf{X}^{l,m}(t+h; t, \mathbf{x}) - \overline{\mathbf{X}^{l,m}}(t+h; t, \mathbf{x}) \\
&= (\mathbf{X}^{l,m}(t+h; t, \mathbf{x}) - \exp(\overline{\mathbf{Z}})\mathbf{x}) - \\
& (\overline{\mathbf{X}^{l,m}}(t+h; t, \mathbf{x}) - \exp(\overline{\mathbf{Z}})\mathbf{x}) \\
&= \sum_{j=2p+1}^{+\infty} c_j \overline{\mathbf{Z}^j} \mathbf{x}, \tag{35}
\end{aligned}$$

where

$$\overline{\mathbf{Z}} = h\mathbf{A}^0 + \sum_{i=1}^s \sqrt{h} \xi_h^i \mathbf{A}^i, \tag{36}$$

$\overline{c_j} j \geq l+m+1$, are constants. Then we obtain

$$\begin{aligned}
& |E(\mathbf{X}^{l,m}(t+h; t, \mathbf{x}) - \overline{\mathbf{X}^{l,m}}(t+h; t, \mathbf{x}))| \\
&\leq \overline{K}_1 \left| E \sum_{j=l+m+1}^{+\infty} (h\mathbf{A}^0 + \sum_{i=1}^s \sqrt{h} \xi_h^i \mathbf{A}^i)^j \right| \\
&= O(h^{\frac{l+m}{2}+1}), \tag{37}
\end{aligned}$$

and

$$\begin{aligned}
& E | \mathbf{X}^{l,m}(t+h; t, \mathbf{x}) - \overline{\mathbf{X}^{l,m}}(t+h; t, \mathbf{x}) |^2 \\
&\leq \overline{K}_2 E \left| \sum_{j=l+m+1}^{+\infty} (h\mathbf{A}^0 + \sum_{i=1}^s \sqrt{h} \xi_h^i \mathbf{A}^i)^j \right|^2 \\
&= O(h^{l+m+1}), \tag{38}
\end{aligned}$$

where \overline{K}_1 and \overline{K}_2 are positive constants independent of h . Applying Proposition 2.2 we complete the proof of Theorem 2.1. \square

3 Preservation of the Poisson structure and the Casimir functions

For the $\overline{\mathbf{Z}}_n$ in (14), it is easy to verify the following propositions:

Proposition 3.1 For any $n \geq 0$, $n \in \mathbb{Z}$,

$$\overline{\mathbf{Z}}_n^{2n} \mathbf{B} = \mathbf{B}(\overline{\mathbf{Z}}_n^{2n})^T,$$

$$\overline{\mathbf{Z}}_n^{2n+1} \mathbf{B} = -\mathbf{B}(\overline{\mathbf{Z}}_n^{2n+1})^T.$$

Proposition 3.2 Suppose $f(x)$ is an even polynomial and $g(x)$ is an odd polynomial, then we have

$$\begin{aligned} f(\bar{\mathbf{Z}}_n)\mathbf{B} &= \mathbf{B}f(\bar{\mathbf{Z}}_n^T), g(\bar{\mathbf{Z}}_n)\mathbf{B} = -\mathbf{B}g(\bar{\mathbf{Z}}_n^T), \\ (f(\bar{\mathbf{Z}}_n) - g(\bar{\mathbf{Z}}_n))(f(\bar{\mathbf{Z}}_n) + g(\bar{\mathbf{Z}}_n))\mathbf{B} \\ &= (f(\bar{\mathbf{Z}}_n) + g(\bar{\mathbf{Z}}_n))(f(\bar{\mathbf{Z}}_n) - g(\bar{\mathbf{Z}}_n))\mathbf{B}. \end{aligned}$$

Now we prove the following theorem:

Theorem 3.1 The (p, p) -schemes (13) ($l = m = p$) preserve the Poisson structure of the original linear stochastic Poisson system (7).

Proof Regarding the equivalent form (11) of (13), we need to verify

$$\begin{aligned} &\frac{\partial \mathbf{X}_{n+1}^{p,p}}{\partial \mathbf{X}_n^{p,p}} \mathbf{B} \left(\frac{\partial \mathbf{X}_{n+1}^{p,p}}{\partial \mathbf{X}_n^{p,p}} \right)^T \\ &= \mathbf{D}_{p,p}^{-1}(\bar{\mathbf{Z}}_n) \mathbf{N}_{p,p}(\bar{\mathbf{Z}}_n) \mathbf{B} \mathbf{N}_{p,p}(\bar{\mathbf{Z}}_n^T) \mathbf{D}_{p,p}^{-1}(\bar{\mathbf{Z}}_n^T) \\ &= \mathbf{B}, \end{aligned} \quad (39)$$

which is equivalent to

$$\mathbf{N}_{p,p}(\bar{\mathbf{Z}}_n) \mathbf{B} \mathbf{N}_{p,p}(\bar{\mathbf{Z}}_n^T) = \mathbf{D}_{p,p}(\bar{\mathbf{Z}}_n) \mathbf{B} \mathbf{D}_{p,p}(\bar{\mathbf{Z}}_n^T). \quad (40)$$

Let

$$\begin{aligned} \mathbf{N}_{p,p}(\bar{\mathbf{Z}}_n) &= f(\bar{\mathbf{Z}}_n) + g(\bar{\mathbf{Z}}_n), \\ \mathbf{D}_{p,p}(\bar{\mathbf{Z}}_n) &= f(\bar{\mathbf{Z}}_n) - g(\bar{\mathbf{Z}}_n), \end{aligned}$$

where $f(x)$ is an even polynomial and $g(x)$ is an odd polynomial. Then it follows

$$\begin{aligned} &\mathbf{N}_{p,p}(\bar{\mathbf{Z}}_n) \mathbf{B} \mathbf{N}_{p,p}(\bar{\mathbf{Z}}_n^T) \\ &= (f(\bar{\mathbf{Z}}_n) + g(\bar{\mathbf{Z}}_n))\mathbf{B}(f(\bar{\mathbf{Z}}_n^T) + g(\bar{\mathbf{Z}}_n^T)) \\ &= (f(\bar{\mathbf{Z}}_n) + g(\bar{\mathbf{Z}}_n))(f(\bar{\mathbf{Z}}_n) - g(\bar{\mathbf{Z}}_n))\mathbf{B} \\ &= (f(\bar{\mathbf{Z}}_n) - g(\bar{\mathbf{Z}}_n))(f(\bar{\mathbf{Z}}_n) + g(\bar{\mathbf{Z}}_n))\mathbf{B} \\ &= (f(\bar{\mathbf{Z}}_n) - g(\bar{\mathbf{Z}}_n))\mathbf{B}(f(\bar{\mathbf{Z}}_n^T) - g(\bar{\mathbf{Z}}_n^T)) \\ &= \mathbf{D}_{p,p}(\bar{\mathbf{Z}}_n) \mathbf{B} \mathbf{D}_{p,p}(\bar{\mathbf{Z}}_n^T). \quad \square \end{aligned}$$

Next we prove the Casimir-preservation by the (p, p) -scheme (13).

Theorem 3.2 The (p, p) -schemes (13) ($l = m = p$) preserve the Casimir functions of the linear stochastic Poisson system (7).

Proof From (13) we know

$$\mathbf{X}_{n+1}^{p,p} - \mathbf{X}_n^{p,p} = \mathbf{B}\boldsymbol{\Theta}_n^{p,p},$$

where

$$\boldsymbol{\Theta}_n^{p,p} = \left(\sum_{k=1}^p n_k^{p,p} \bar{\mathbf{Y}}_n (\mathbf{B} \bar{\mathbf{Y}}_n)^{k-1} \right) \mathbf{X}_n^{p,p} -$$

$$\left(\sum_{k=1}^p d_k^{p,p} (-1)^k \bar{\mathbf{Y}}_n (\mathbf{B} \bar{\mathbf{Y}}_n)^{k-1} \right) \mathbf{X}_{n+1}^{p,p},$$

with $\bar{\mathbf{Y}}_n = h\mathbf{C}^0 + \sum_{i=1}^s \sqrt{h} \boldsymbol{\zeta}_{n,h}^i \mathbf{C}^i$. By the fundamental theorem of calculus, we have, for any Casimir function $C(\mathbf{X})$ of the linear stochastic Poisson system (7),

$$\begin{aligned} &C(\mathbf{X}_{n+1}^{p,p}) - C(\mathbf{X}_n^{p,p}) \\ &= \int_0^1 \nabla C(\mathbf{X}_n^{p,p} + \tau(\mathbf{X}_{n+1}^{p,p} - \mathbf{X}_n^{p,p}))^T (\mathbf{X}_{n+1}^{p,p} - \mathbf{X}_n^{p,p}) d\tau \\ &= \int_0^1 \nabla C(\mathbf{X}_n^{p,p} + \tau(\mathbf{X}_{n+1}^{p,p} - \mathbf{X}_n^{p,p}))^T \mathbf{B} \boldsymbol{\Theta}_n^{p,p} d\tau \\ &= 0, \end{aligned}$$

where the last step is due to the fact that $C(\mathbf{X})$ is a Casimir function such that $\nabla C(\mathbf{X})^T \mathbf{B} \equiv 0$, for all \mathbf{X} . \square

4 Numerical experiments

In this section, we illustrate the performance of the (p, p) -schemes (13) ($l = m = p$) via numerical tests. We consider the following linear stochastic Poisson system

$$\begin{aligned} d\mathbf{X}(t) &= \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 3 \\ 1 & -3 & 0 \end{pmatrix} \cdot \left[\begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \mathbf{X} dt + \right. \\ &\quad \left. \frac{1}{4} \begin{pmatrix} 11 & 4 & 4 \\ 4 & 2 & 1 \\ 4 & 1 & 2 \end{pmatrix} \mathbf{X} \circ d\mathbf{W}(t) \right], \end{aligned}$$

$$\mathbf{X}(t_0) = \mathbf{x}. \quad (41)$$

It can be proved that, the function $C(\mathbf{X}) = 3\mathbf{X}^{(1)} + \mathbf{X}^{(2)} + \mathbf{X}^{(3)}$ is a Casimir function of the system (41). Moreover, the Hamiltonian functions

$$\begin{aligned} H^1(\mathbf{X}) &= (\mathbf{X}^{(1)} + \mathbf{X}^{(2)})^2 + (\mathbf{X}^{(1)} + \mathbf{X}^{(3)})^2, \\ H^2(\mathbf{X}) &= 11(\mathbf{X}^{(1)})^2 + 2(\mathbf{X}^{(2)})^2 + 2(\mathbf{X}^{(3)})^2 + \\ &\quad 8\mathbf{X}^{(1)}\mathbf{X}^{(2)} + 2\mathbf{X}^{(2)}\mathbf{X}^{(3)} + 8\mathbf{X}^{(3)}\mathbf{X}^{(1)} \end{aligned}$$

are invariants of the system (41), since it holds

$$(\nabla H^1)^T \mathbf{B} \nabla H^2 = 0, \text{ where } \mathbf{B} = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 3 \\ 1 & -3 & 0 \end{pmatrix}.$$

The stochastic poisson integrators, i. e., the (p, p) -schemes (13) ($p=1, 2, 3$) for (41) read

$$\mathbf{X}_{n+1}^{1,1} = \left(\mathbf{I}_3 + \frac{1}{2} \bar{\mathbf{Z}}_n \right) \mathbf{X}_n^{1,1} + \frac{1}{2} \bar{\mathbf{Z}}_n \mathbf{X}_{n+1}^{1,1}, \quad (42)$$

$$\begin{aligned} X_{n+1}^{2,2} &= \left(I_3 + \frac{1}{2} \bar{Z}_n + \frac{1}{12} (\bar{Z}_n)^2 \right) X_n^{2,2} + \\ &\quad \left(\frac{1}{2} \bar{Z}_n - \frac{1}{12} (\bar{Z}_n)^2 \right) X_{n+1}^{2,2}, \quad (43) \\ X_{n+1}^{3,3} &= \left(I_3 + \frac{1}{2} \bar{Z}_n + \frac{1}{10} (\bar{Z}_n)^2 + \frac{1}{120} (\bar{Z}_n)^3 \right) X_n^{3,3} + \\ &\quad \left(\frac{1}{2} \bar{Z}_n - \frac{1}{10} (\bar{Z}_n)^2 + \frac{1}{120} (\bar{Z}_n)^3 \right) X_{n+1}^{3,3}, \quad (44) \end{aligned}$$

where

$$\begin{aligned} A^0 &= \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 3 \\ 1 & -3 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \\ A^1 &= \frac{1}{4} \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 3 \\ 1 & -3 & 0 \end{pmatrix} \begin{pmatrix} 11 & 4 & 4 \\ 4 & 2 & 1 \\ 4 & 1 & 2 \end{pmatrix}, \\ \bar{Z}_n &= hA^0 + \sqrt{h}\zeta_{n,h}^1 A^1, \end{aligned}$$

with $A_h = \sqrt{4p \mid \ln h \mid}$, $p = 1, 2, 3$.

Figure 1 illustrates the root mean-square orders of the (p, p) -schemes (42), (43) and (44), which are 1, 2 and 3, respectively, coinciding with the theoretical result in Theorem 2.1. Here we take $T = 10$, initial value $\mathbf{x} = [1, 0, -1]$, and time steps $\mathbf{h} = [0.005, 0.01, 0.02, 0.025, 0.05]$. The expectation is approximated by taking average over 1 000 sample paths.

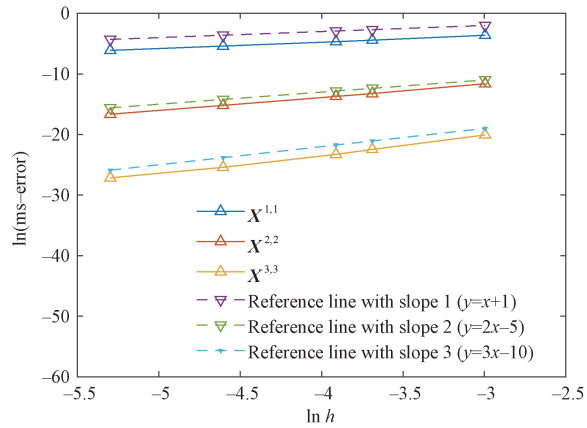


Fig. 1 Root mean-square convergence orders of (42), (43), and (44)

Figure 2 compares the numerical sample paths of $X^{(1)}(t)$, $X^{(2)}(t)$ and $X^{(3)}(t)$ arising from the $(1,1)$ -scheme (42) and the implicit Euler method with the reference solution. As can be seen, as time step $h = 0.01$, both numerical methods can create good path approximations. As $h = 0.1$, however, the

implicit Euler fails to simulate the paths in acceptable accuracy, while our $(1,1)$ -scheme (42) still behaves fairly well.

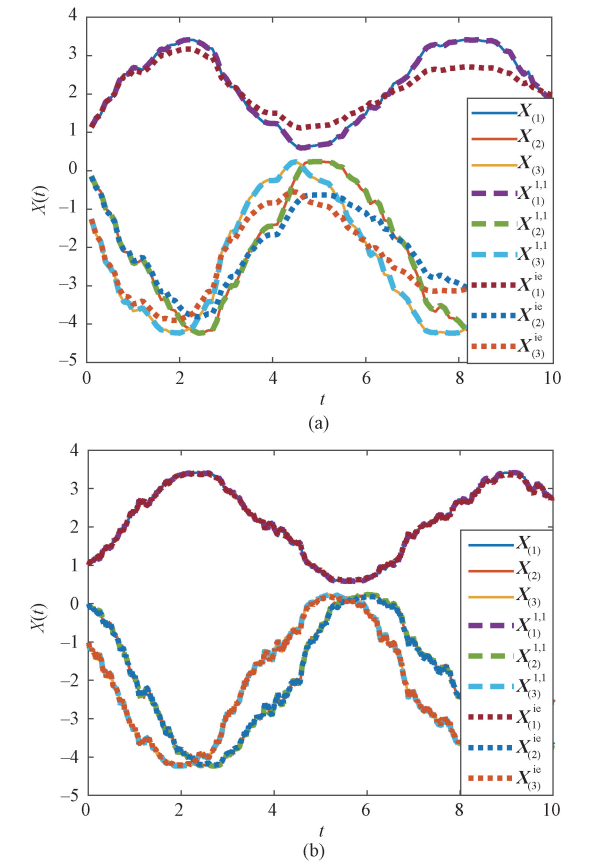


Fig. 2 Sample paths produced by (42) and the implicit Euler method for $h=0.1$ (a) and $h=0.01$ (b)

Figure 3 displays the computed Casimir function $C(X_n) = 3X_n^{(1)} + X_n^{(2)} + X_n^{(3)}$ arising from the $(1,1)$ -scheme (42) and the implicit Euler method with initial $\mathbf{x} = [1, 1, 1]$, from the $(2,2)$ -scheme (43) with $\mathbf{x} = [1, 1, 2]$, and from the $(3,3)$ -scheme (44) with $\mathbf{x} = [1, 1, 3]$. We take $T = 10, h = 0.1$. Obviously, all the methods can preserve the Casimir function.

Figure 4 observes the evolution of the Hamiltonians $H^1(X(t))$ and $H^2(X(t))$ over the time interval $t \in [0, 10]$, produced by the $(1,1)$ -scheme (42) and the implicit Euler with initial $\mathbf{x} = [1, 1, 1]$, by the $(2,2)$ -scheme (43) with initial $\mathbf{x} = [1, 1, 2]$ and by the $(3,3)$ -scheme (44) with initial $\mathbf{x} = [1, 1, 3]$. It is clear that, the implicit Euler can not preserve the Hamiltonians, while our (p, p) -schemes ($p = 1, 2, 3$) can. Here we take $h = 0.1$.

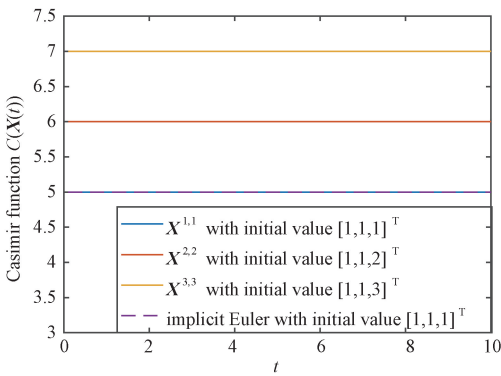


Fig. 3 Preservation of Casimir function by (42), (43), (44), and the implicit Euler method

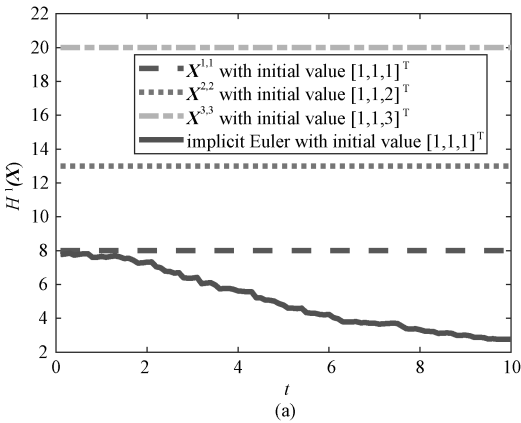


Figure 5 demonstrates the evolution of the relative error $\frac{\left| \frac{\partial X_{n+1}}{\partial X_n} B \left(\frac{\partial X_{n+1}}{\partial X_n} \right)^T - B \right|}{|B|}$, arising from the (p,p) -schemes (42), (43), (44), and the implicit Euler method, which indicates the ability of preserving the Poisson structure by the methods. As shown by the figure, the implicit Euler method can not preserve the Poisson structure exactly, while our (p,p) -schemes can. Here the time step is $h = 0.01$.

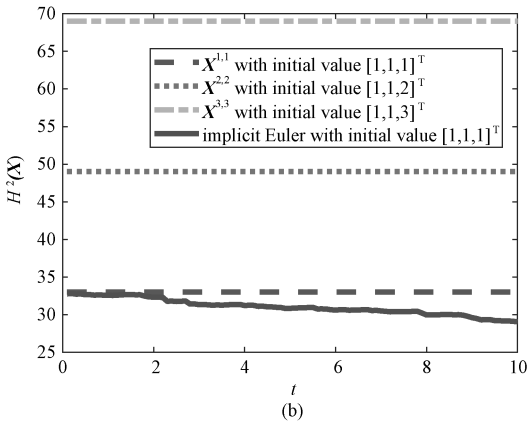


Fig. 4 Evolutions of H^1 (a) and H^2 (b) by (42), (43), (44), and the implicit Euler method

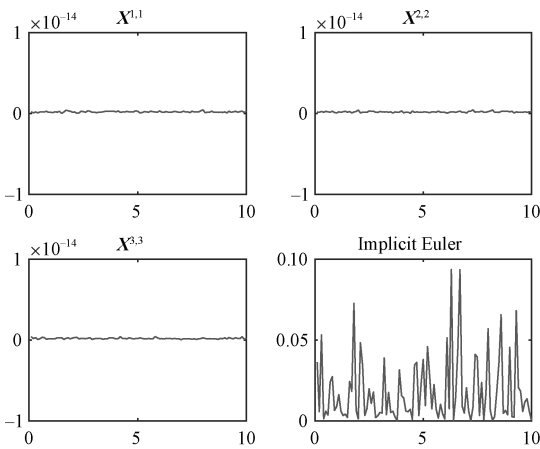


Fig. 5 Error evolutions of the Poisson structure by (42), (43), (44), and the implicit Euler method

5 Conclusion

Theoretical and empirical analyses show that the (p,p) -schemes, based on the Padé approximations, can simulate the linear stochastic Poisson systems with arbitrarily high root mean-square orders p ($p \geq 1, p \in \mathbb{Z}$), and preservation

of both the Poisson structure and the Casimir functions. They constitute a class of stochastic Poisson integrators for linear stochastic Poisson systems, whose superiorities to non-Poisson integrators are illustrated by numerical experiments.

References

[1] Hong J L, Ruan J L, Sun L Y, et al. Structure-preserving numerical methods for stochastic Poisson systems [J]. Communications in Computational Physics, 2021, 29 (3) : 802-830.

[2] Milstein G N, Repin Y M, Tretyakov M V. Symplectic integration of Hamiltonian systems with additive noise [J]. SIAM Journal on Numerical Analysis, 2002, 39 (6) : 2066-2088.

[3] Milstein G N, Repin Y M, Tretyakov M V. Numerical methods for stochastic systems preserving symplectic structure [J]. SIAM Journal on Numerical Analysis, 2002, 40 (4) : 1583-1604.

[4] Ruan J L, Wang L J. A Lie algebraic approach for a class of highly oscillatory stochastic Hamiltonian systems [J]. Journal of University of Chinese Academy of Sciences, 2019, 36 (1) : 5-10.

- [5] Feng K, Qin M Z. Symplectic geometric algorithms for Hamiltonian systems [M]. Berlin: Springer Science & Business Media, 2010.
- [6] Hairer E, Lubich C, Wanner G. Geometric numerical integration[M]. Berlin: Springer Science & Business Media, 2006.
- [7] Zhu W J, Qin M Z. Poisson schemes for Hamiltonian systems on Poisson manifolds [J]. Computers & Mathematics with Applications, 1994, 27(12): 7-16.
- [8] Cohen D, Dujardin G. Energy-preserving integrators for stochastic Poisson systems [J]. Communications in Mathematical Sciences, 2014, 12(8): 1523-1539.
- [9] Hong J L, Ji L H, Wang X. Stochastic K-symplectic integrators for stochastic non-canonical Hamiltonian systems and applications to the Lotka-Volterra model [EB/OL]. arXiv:1711.03258v1 [math. NA] (2017-11-09) [2019-07-02]. <https://arxiv.org/abs/1711.03258>.
- [10] Feng K, Wu H M, Qin M Z. Symplectic difference schemes for linear Hamiltonian canonical systems [J]. Journal of Computational Mathematics, 1990, 8(4): 371-380.
- [11] Sun L Y, Wang L J. Stochastic symplectic methods based on the Padé approximations for linear stochastic Hamiltonian systems [J]. Journal of Computational and Applied Mathematics, 2017, 311: 439-456.
- [12] Aboanber A E, Nahla A A. On Padé approximations to the exponential function and application to the point kinetics equations[J]. Progress in Nuclear Energy, 2004, 44(4): 347-368.
- [13] Milstein G N. Numerical integration of stochastic differential equations[M]. Berlin: Springer Science & Business Media, 1995.