文章编号:2095-6134(2023)01-0001-05

# Limiting property of distribution function in Lorentz space*

WU Di[1], DENG Yangkendi[2], YU Dandan[2], YAN Dunyan[2†]

(1 *School of Science*, *Zhejiang University of Science and Technology*, *Hangzhou* 310023, *China*;
2 *School of Mathematical Sciences*, *University of Chinese Academy of Sciences*, *Beijing* 100049, *China*)

(Received 5 January 2021; Revised 3 March 2021)

**Wu D, Deng Y K D, Yu D D, et al. Limiting property of distribution function in Lorentz space[J]. Journal of University of Chinese Academy of Sciences, 2023, 40(1): 1-5. DOI:10.7523/j.ucas.2021.0015.**

**Abstract**　In this paper, we give a novel proof for the following equality

$$\lim_{\alpha\to 0^+}\alpha^p d_f(\alpha)=\lim_{\alpha\to\infty}\alpha^p d_f(\alpha)=0$$

for $f\in L^{p,q}(X,\mu)$ with $0<p<\infty$, and $0<q<\infty$. We also prove that the function $\alpha^p$ can not be improved for some sense. When $q=\infty$, the above equality does not hold.

**Keywords**　limiting behavior; distribution functions; Lorentz spaces

**CLC number**: O177　**Document code**: A　**DOI**: 10.7523/j.ucas.2021.0015

# 洛伦兹空间上的分布函数的极限性质

吴迪[1],邓杨肯迪[2],于丹丹[2],燕敦验[2]

(1 浙江科技学院理学院, 杭州 310023; 2 中国科学院大学数学科学学院, 北京 100049)

**摘　要**　用一个新颖的方法证明以下等式:

$$\lim_{\alpha\to 0^+}\alpha^p d_f(\alpha)=\lim_{\alpha\to\infty}\alpha^p d_f(\alpha)=0$$

其中$f\in L^{p,q}(X,\mu)$,并且有$0<p<\infty$和$0<q<\infty$。也证明函数$\alpha^p$在某种意义下不能再提升。特别地,当$q=\infty$时,以上等式是不一定成立的。

**关键词**　极限行为;分布函数;洛伦兹空间

Let $f$ be a measurable function on a measure space $(X,\mu)$ and $0<p<\infty$, $0<q\leqslant\infty$. Define

$$\|f\|_{L^{p,q}}=\begin{cases}p^{\frac{1}{q}}\left(\int_0^\infty (d_f(\alpha)^{\frac{1}{p}}\alpha)^q\frac{\mathrm{d}\alpha}{\alpha}\right)^{\frac{1}{q}}, & \text{if } q<\infty,\\ \sup\limits_{\alpha>0}\alpha\, d_f(\alpha)^{\frac{1}{p}}, & \text{if } q=\infty,\end{cases}\tag{1}$$

where $d_f(\alpha)=\mu\{x:|f(x)|>\alpha\}$ is the distribution function of $f$. The set of all $f$ with $|f|_{L^{p,q}}<\infty$ is denoted by $L^{p,q}(X,\mu)$ and is called the Lorentz space with indices $p$ and $q$.

There are many simple properties of Lorentz space. For $0<p<\infty$, we have that

$$L^{p,p}(X,\mu)=L^p(X,\mu),$$

* Supported by NSF of Zhejiang Province of China (LQ18A010002, LQ17A010002)

† Corresponding author, E-mail: ydunyan@ucas.ac.cn

and

$$L^{p,\infty}(X,\mu)=WL^p(X,\mu).$$

For $0<p<\infty$ and $0<q<r\leqslant\infty$, there exists a constant $c_{p,q,r}$ such that

$$\|f\|_{L^{p,r}}\leqslant c_{p,q,r}\|f\|_{L^{p,q}}.$$

More properties of Lorentz space can be found in Refs. [1-4].

The limiting property of distribution functions of $L^p$ functions has been proved in Ref. [5] for $1\leqslant p<\infty$. That is, when $f\in L^{p,p}(\mathbb{R}^n)$, we have that

$$\lim_{\alpha\to 0^+}\alpha^p d_f(\alpha)=0. \tag{2}$$

However, for $f\in L^1(\mathbb{R}^n)$, we know that $Mf\in L^{1,\infty}(\mathbb{R}^n)$, where $M$ is the maximal operator. In Ref. [6], P. Janakiraman has proved that

$$\lim_{\alpha\to 0^+}\alpha d_{Mf}(\alpha)=\|f\|_{L^1(\mathbb{R}^n)}.$$

This means that the limiting equality of distribution functions does not hold for $L^{1,\infty}$ function.

It is well-known that the following inclusion relation

$$L^{p,p}(\mathbb{R}^n)\subsetneqq L^{p,q}(\mathbb{R}^n)\subsetneqq L^{p,\infty}(\mathbb{R}^n)$$

holds for $0<p<q<\infty$.

It is worth investigating the limiting property of distribution functions of $f\in L^{p,q}(X,\mu)$.

# 1 Main results and proof

Now we formulate our main theorem.

**Theorem 1.1** Let $\mu$ be a $\sigma$-finite positive measure on some $\sigma$-algebra in set $X$. For $f\in L^{p,q}(X,\mu)$ with $0<p,q<\infty$, denote the distribution functions of $f$ as

$$d_f(\alpha):=\mu(\{x\in X:|f(x)|>\alpha\}).$$

Then we have

$$\lim_{\alpha\to 0^+}\alpha^p d_f(\alpha)=\lim_{\alpha\to\infty}\alpha^p d_f(\alpha)=0. \tag{3}$$

**Proof** First, we prove that

$$\limsup_{\alpha\to 0^+}\alpha^p d_f(\alpha)=0.$$

If the conclusion does not hold, then we conclude that

$$\limsup_{\alpha\to 0^+}\alpha^p d_f(\alpha)=C_0>0,$$

or

$$\limsup_{\alpha\to 0^+}\alpha^p d_f(\alpha)=\infty.$$

Thus there exist a positive sequence $\{x_n\}_{n=1}^\infty$ and a positive number $C$ such that

$$\lim_{n\to\infty}x_n=0,$$

and

$$x_n^p d_f(x_n)\geqslant C,$$

for $n\in\mathbb{N}$. Obviously, there exists a strictly decreasing subsequence, still denoted as $\{x_n\}$.

Since $d_f$ is a decreasing function, it follows that

$$\begin{aligned}
\|f\|_{L^{p,q}}^q&=p\int_0^\infty\left(\alpha d_f(\alpha)^{\frac{1}{p}}\right)^q\frac{\mathrm{d}\alpha}{\alpha}\\
&=p\int_0^\infty\alpha^{q-1}d_f(\alpha)^{\frac{q}{p}}\mathrm{d}\alpha\\
&=p\int_0^{x_1}\alpha^{q-1}d_f(\alpha)^{\frac{q}{p}}\mathrm{d}\alpha+p\int_{x_1}^\infty\alpha^{q-1}d_f(\alpha)^{\frac{q}{p}}\mathrm{d}\alpha\\
&\geqslant p\sum_{n=1}^\infty\int_{x_{n+1}}^{x_n}\alpha^{q-1}d_f(\alpha)^{\frac{q}{p}}\mathrm{d}\alpha\\
&\geqslant p\sum_{n=1}^\infty\int_{x_{n+1}}^{x_n}\alpha^{q-1}d_f(x_n)^{\frac{q}{p}}\mathrm{d}\alpha\\
&=\frac{p}{q}\sum_{n=1}^\infty(x_n^q-x_{n+1}^q)d_f(x_n)^{\frac{q}{p}}\\
&\geqslant\frac{pC^{\frac{q}{p}}}{q}\sum_{n=1}^\infty\left(1-\left(\frac{x_{n+1}}{x_n}\right)^q\right).
\end{aligned}\tag{4}$$

Setting $\lambda_n=(x_{n+1}/x_n)^q\in(0,1)$, the basic principle of mathematics analysis tells us that the infinite product $\prod\lambda_n$ converges to a nonzero number if and only if $\sum(1-\lambda_n)$ converges. However we deduce that

$$\begin{aligned}
\prod_{n=1}^\infty\lambda_n&=\lim_{N\to\infty}\prod_{n=1}^N\lambda_n=\lim_{N\to\infty}\prod_{n=1}^N\left(\frac{x_{n+1}}{x_n}\right)^q\\
&=\lim_{N\to\infty}\left(\frac{x_{N+1}}{x_1}\right)^q=0.
\end{aligned}$$

It implies from (4) that

$$\|f\|_{L^{p,q}}=\infty.$$

This leads to contradiction.

Next, we prove

$$\limsup_{\alpha\to\infty}\alpha^p d_f(\alpha)=0.$$

Similarly, if the upper limit is a positive number or $\infty$, there exist a positive number $C$ and a strictly increasing sequence $\{x_n\}_{n=1}^\infty$ such that

$$\lim_{n\to\infty}x_n=\infty$$

and

$$x_n^p d_f(x_n)\geqslant C.$$

Therefore, we conclude that

$$
\begin{aligned}
\|f\|_{L^{p,q}}^{q} &= p\int_0^\infty (\alpha d_f(\alpha)^{\frac{1}{p}})^q \frac{d\alpha}{\alpha} \\
&= p\int_0^\infty \alpha^{q-1} d_f(\alpha)^{\frac{q}{p}} d\alpha \\
&= p\int_0^{x_1} \alpha^{q-1} d_f(\alpha)^{\frac{q}{p}} d\alpha + p\int_{x_1}^\infty \alpha^{q-1} d_f(\alpha)^{\frac{q}{p}} d\alpha \\
&\geqslant p\sum_{n=1}^\infty \int_{x_n}^{x_{n+1}} \alpha^{q-1} d_f(\alpha)^{\frac{q}{p}} d\alpha \\
&\geqslant p\sum_{n=1}^\infty \int_{x_n}^{x_{n+1}} \alpha^{q-1} d_f(x_{n+1})^{\frac{q}{p}} d\alpha \\
&= \frac{p}{q}\sum_{n=1}^\infty (x_{n+1}^q - x_n^q) d_f(x_{n+1})^{\frac{q}{p}} \\
&\geqslant \frac{pC^{q/p}}{q}\sum_{n=1}^\infty \left(1 - \left(\frac{x_n}{x_{n+1}}\right)^q\right). \qquad (5)
\end{aligned}
$$

Note that the infinite product $\prod (x_n/x_{n+1})^q$ diverges to 0. For $\alpha > 0$, $\alpha^p d_f(\alpha) \geqslant 0$ implies

$$\liminf_{\alpha\to\infty} \alpha^p d_f(\alpha) \geqslant 0.$$

That leads to (3). □

Now, we begin to prove that the function $\alpha^p$ in (2) can not be improved. For this purpose, we first give the following lemma which characterize the special property of some function.

**Lemma 1.1** Let $g:(0, +\infty) \to [0, +\infty)$ be a right-continuous non-increasing function. Then there exists a function $f:\mathbb{R}^n \to [0, +\infty]$ such that $d_f(\alpha) = g(\alpha)$ for any $\alpha \in (0, +\infty)$.

**Proof** Let

$$f(x) = \int_0^\infty \chi_{B(0,(\frac{g(u)}{v_n})^{\frac{1}{n}})}(x)\,du, \qquad (6)$$

where $v_n = m(B(0,1))$ and $B(x,r)$ denotes a ball with the center at $x$ and the radius $r$.

Then we merely need to prove that

$$\{x: f(x) > \alpha\} = \left\{x: |x| < \left(\frac{g(\alpha)}{v_n}\right)^{\frac{1}{n}}\right\} \qquad (7)$$

holds, for any $\alpha \in (0, +\infty)$.

Assume

$$|x| \geqslant \left(\frac{g(\alpha)}{v_n}\right)^{\frac{1}{n}}.$$

Note that $g$ is a non-increasing function. We conclude from definition of $f$ in (6) that

$$
\begin{aligned}
f(x) &= \int_0^\alpha \chi_{B(0,(\frac{g(u)}{v_n})^{\frac{1}{n}})}(x)\,du + \int_\alpha^\infty \chi_{B(0,(\frac{g(u)}{v_n})^{\frac{1}{n}})}(x)\,du \\
&= \int_0^\alpha \chi_{B(0,(\frac{g(u)}{v_n})^{\frac{1}{n}})}(x)\,du \\
&\leqslant \alpha. \qquad (8)
\end{aligned}
$$

Thus the inequality (8) implies that the set of the left hand of (7) is contained in the right hand.

Assume

$$|x| < \left(\frac{g(\alpha)}{v_n}\right)^{\frac{1}{n}}.$$

Note that $g$ is a right-continuous function. Then there exists $\alpha' > \alpha$ such that

$$|x| < \left(\frac{g(\alpha')}{v_n}\right)^{\frac{1}{n}}.$$

Then we conclude that

$$
\begin{aligned}
f(x) &= \int_0^{\alpha'} \chi_{B(0,(\frac{g(u)}{v_n})^{\frac{1}{n}})}(x)\,du + \int_{\alpha'}^\infty \chi_{B(0,(\frac{g(u)}{v_n})^{\frac{1}{n}})}(x)\,du \\
&\geqslant \int_0^{\alpha'} \chi_{B(0,(\frac{g(u)}{v_n})^{\frac{1}{n}})}(x)\,du \\
&= \alpha' > \alpha.
\end{aligned}
$$

This means that

$$\{x: f(x) > \alpha\} \supseteqq \left\{x: |x| < \left(\frac{g(\alpha)}{v_n}\right)^{\frac{1}{n}}\right\}.$$

In a word, (7) holds. □

As an application of Lemma 1.2, we prove that the function $\alpha^p$ can not be improved for some sense.

**Theorem 1.2** Suppose $0 < p,q < \infty$. For any nonnegative function $h$ satisfying

$$\lim_{\alpha\to 0^+} h(\alpha) = \infty, \qquad (9)$$

there exists a function $f \in L^{p,q}(\mathbb{R}^n)$ such that

$$\limsup_{\alpha\to 0^+} \alpha^p h(\alpha) d_f(\alpha) = \infty.$$

**Proof** If $x_k > 0$ and $x_k \to 0$, by (9), then there exists a strictly decreasing positive subsequence, still denoted by $\{x_k\}_{k=1}^\infty$, such that

$$h(\alpha_k) \geqslant 4^k$$

and

$$x_{k+1} < \min\{2^{-1/p}, 2^{-1}\} x_k,$$

for $k \in \mathbb{N}$ and $\alpha_k \in [2^{-1}x_k, x_k)$.

Define a function $g:(0, +\infty) \to [0, +\infty)$ as

$$g(\alpha) = \begin{cases} \dfrac{1}{2^k x_k^p}, & \text{if } x_{k+1} \leqslant \alpha < x_k, \\ 0, & \text{if } \alpha \geqslant x_1. \end{cases} \qquad (10)$$

By Lemma 1.1, there exists a function $f:\mathbb{R}^n \to [0,+\infty]$ such that $d_f(\alpha)=g(\alpha)$ for any $\alpha \in (0,+\infty)$. And we have from the definition of $g$ in (10) that

$$\begin{aligned}\|f\|_{L^{p,q}}^q &= p\int_0^\infty \alpha^{q-1} d_f(\alpha)^{q/p}\mathrm{d}\alpha \\ &= p\sum_{k=1}^\infty \int_{x_{k+1}}^{x_k} \alpha^{q-1} d_f(\alpha)^{q/p}\mathrm{d}\alpha \\ &= \frac{p}{q}\sum_{k=1}^\infty \left(1-\left(\frac{x_{k+1}}{x_k}\right)^q\right)2^{-\frac{qk}{p}} \\ &\leqslant \frac{p}{q}\sum_{k=1}^\infty 2^{-\frac{qk}{p}} < \infty.\end{aligned}$$

This implies $f \in L^{p,q}(\mathbb{R}^n)$.

On the other hand, it follows that

$$\alpha_k^p h(\alpha_k) d_f(\alpha_k) \geqslant \left(\frac{x_k}{2}\right)^p 4^k \frac{1}{2^k x_k^p} = 2^{k-p}.$$

Consequently, we obtain

$$\limsup_{\alpha\to 0^+} \alpha^p h(\alpha) d_f(\alpha) = \infty. \qquad \square$$

**Remark** In the fact, we can also prove that, for any $h$ satisfying

$$\lim_{\alpha\to\infty} h(\alpha) = \infty,$$

there exists a function $f \in L^{p,q}(\mathbb{R}^n)$ such that

$$\limsup_{\alpha\to\infty} \alpha^p h(\alpha) d_f(\alpha) = \infty,$$

where $0<p,q<\infty$. The case can be proved similarly so we omit the details.

Theorem 1.2 tells us that the function $\alpha^p$ in (3) can not be improved. For $h(\alpha)=|\log\alpha|^s$ where $s>0$, we have some more explicit conclusions.

**Corollary 1.1** Suppose $s$ is a positive constant and $0<p,q<\infty$. If $s>p/q$, then there exists a function $f\in L^{p,q}(\mathbb{R}^n)$ such that

$$\lim_{\alpha\to 0^+} \alpha^p |\log\alpha|^s d_f(\alpha) = \infty;$$

if $0<s\leqslant p/q$, then for any $f\in L^{p,q}(\mathbb{R}^n)$, we have

$$\liminf_{\alpha\to 0^+} \alpha^p |\log\alpha|^s d_f(\alpha) = 0. \qquad (11)$$

**Proof** First, we assume that $s>p/q$. Let $g:(0,+\infty)\to[0,+\infty)$ be defined by

$$g(\alpha) = \chi_{(0,a)}(\alpha)\frac{1}{\alpha^p|\log\alpha|^{\frac{1}{2}(s+p/q)}},$$

where $a$ is a sufficiently small positive constant such that $g(\alpha)$ is non-increasing.

By Lemma 1.1, there exists a function $f:\mathbb{R}^n\to[0,+\infty]$ such that $d_f(\alpha)=g(\alpha)$ for any $\alpha\in(0,+\infty)$.

In fact, we can choose the function $f$ as in (6). It implies from simple calculation that

$$f\in L^{p,q}(\mathbb{R}^n)$$

and

$$\lim_{\alpha\to 0^+} \alpha^p |\log\alpha|^s d_f(\alpha) = \infty.$$

Next, we assume that $0<s\leqslant p/q$. If the limit in (11) does not hold, then there exists a function $f\in L^{p,q}(\mathbb{R}^n)$ satisfying

$$\liminf_{\alpha\to 0^+} \alpha^p |\log\alpha|^s d_f(\alpha) = C_0 > 0,$$

or

$$\liminf_{\alpha\to 0^+} \alpha^p |\log\alpha|^s d_f(\alpha) = \infty.$$

Thus there exists constants $C>0$ and $0<a<1$, such that for $\alpha<a$,

$$\alpha^p |\log\alpha|^s d_f(\alpha) \geqslant C.$$

Observe that $0<s\leqslant p/q$. We have that

$$\begin{aligned}\|f\|_{L^{p,q}}^q &= p\int_0^\infty (\alpha d_f(\alpha)^{\frac{1}{p}})^q \frac{\mathrm{d}\alpha}{\alpha} \\ &\geqslant p\int_0^a (\alpha d_f(\alpha)^{\frac{1}{p}})^q \frac{\mathrm{d}\alpha}{\alpha} \\ &\geqslant p\int_0^a \left(\alpha\left(\frac{C}{\alpha^p|\log\alpha|^s}\right)^{\frac{1}{p}}\right)^q\frac{\mathrm{d}\alpha}{\alpha} \\ &= pC^{q/p}\int_0^a \frac{1}{\alpha|\log\alpha|^{sq/p}}\mathrm{d}\alpha \\ &= \infty.\end{aligned}$$

This leads to a contradiction with $f\in L^{p,q}(\mathbb{R}^n)$. $\square$

**Remark** By Corollary 1.1, we know that upper limit can not be replaced into limit in Theorem 1.2.

The following corollary tells us the limiting property does not hold for $L^{p,\infty}$ function.

**Corollary 1.2** Let $0<p<\infty$. For a fixed constant $C>0$, there exists a function $f\in L^{p,\infty}(\mathbb{R}^n)$ such that

$$\lim_{\alpha\to 0^+}\alpha^p d_f(\alpha) = \lim_{\alpha\to\infty}\alpha^p d_f(\alpha) = C.$$

**Proof** Let $g:(0,+\infty)\to[0,+\infty)$ be defined by

$$g(\alpha) = \frac{C}{\alpha^p}.$$

By Lemma 1.1, there exists a function $f:\mathbb{R}^n\to[0,+\infty]$ such that $d_f(\alpha)=g(\alpha)$ for any $\alpha\in(0,+\infty)$.

We can easily obtain that

$$f\in L^{p,\infty}(\mathbb{R}^n)$$

and

$$\lim_{\alpha \to 0^+} \alpha^p d_f(\alpha) = \lim_{\alpha \to \infty} \alpha^p d_f(\alpha) = C. \qquad \square$$

## References

[1] Grafakos L. Classical and modern Fourier analysis [M]. Upper Saddle River, NJ: Pearson/Prentice Hall, 2004.

[2] Lu S Z, Ding Y, Yan D Y. Singular integral and related topics [M]. Singapore: World Scientific, 2007.

[3] Wiener N. The ergodic theorem [J]. Duke Mathematical Journal, 1939, 5(1): 1-18. DOI: 10.1215/S0012-7094-39-00501-6.

[4] Wheeden R L, Zygmund A. Measure and integral [M]. 2nd ed. Boca Raton, Florida: CRC Press, 2015.

[5] Xu S Z, Yan D Y. Limiting property of the distribution function of $L^p$ function at endpoints [J]. Journal of Mathematical Research with Applications, 2016, 36(2): 177-182.

[6] Janakiraman P. Limiting weak-type behavior for singular integral and maximal operators [J]. Transactions of the American Mathematical Society, 2006, 358(5): 1937-1952. DOI: 10.1090/s0002-9947-05-04097-3.