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Construction of totally real surfaces in complex Grassmannians*

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Abstract We present a construction of the complex Grassmannian $G(2,n+2)$ as a quotient of some minimal submanifold Q^{n+1} of $\mathbb{H}P^{n+1}$, then show that a surface in $G(2,n+2)$ can be horizontally lifted to Q^{n+1} if and only if it is totally real.

Keywords Grassmannian; totally real surface; horizontal lift

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复 Grassmann 流形中全实曲面的构造

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摘 要 给出复 Grassmann 流形 $G(2,n+2)$ 的全实曲面的一种构造方法,也就是把 $G(2,n+2)$ 看作 $\mathbb{H}P^{n+1}$ 中极小子流形 Q^{n+1} 的商,并证明 $G(2,n+2)$ 中的曲面可以水平提升到 Q^{n+1} 中当且仅当它是全实的。

关键词 Grassmann 流形;全实曲面;水平提升

The theory of minimal surfaces is an important part of modern differential geometry. The theory is particularly fruitful when the ambient space is a symmetric space. Calabi[1] proved a rigidity theorem for minimal two-spheres of constant curvature in S^n. Bolton et al.[2] constructed all the minimal two-spheres of constant curvature in $\mathbb{C}P^n$, and showed that a totally real minimal two-sphere in $\mathbb{C}P^n$ can be mapped, by a holomorphic isometry of $\mathbb{C}P^n$, into $\mathbb{R}P^n \subset \mathbb{C}P^n$. Then He and Wang[3] proved a similar rigidity result for totally real minimal two-spheres in $\mathbb{H}P^n$.

In this paper, we present a construction of the complex Grassmannian $G(2,n+2)$ due to Berndt[4], which considers $G(2,n+2)$ as a quotient of some minimal submanifold Q^{n+1} of $\mathbb{H}P^{n+1}$. A Riemannian metric can be given on $G(2,n+2)$ so that the projection $\pi: Q^{n+1} \to G(2,n+2)$ is a Riemannian submersion. Then we show that a surface in $G(2,n+2)$ can be horizontally lifted to Q^{n+1} if and only if it

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is totally real.

Our result is a special case of Ref. [5], where the author considered a general Riemannian submersion $N \to B$, and characterized the existence of horizontal lifts of a submanifold of B using a family J of $(1,1)$-tensors on B. In our paper, we make use of the fact that the projection $Q^{n+1} \to G(2,n+2)$ is a principal bundle, thus obtain a characterization by a first order PDE. Our method is largely inspired by Ref. [3], where the authors considered the Riemannian submersion $S^{4n+3} \to \mathbb{H}P^n$.

1 Preliminaries

We denote by $\mathbb{H}$ the algebra of quaternions. This is a 4-dimensional vector space over $\mathbb{R}$ with basis $\{1, \mathrm{i}, \mathrm{j}, \mathrm{k}\}$, and the multiplication is defined by

$$\mathrm{i}^2 = \mathrm{j}^2 = \mathrm{k}^2 = -1,$$

$$\mathrm{ij} = \mathrm{k} = -\mathrm{ji}, \mathrm{jk} = \mathrm{i} = -\mathrm{kj}, \mathrm{ki} = \mathrm{j} = -\mathrm{ik}.$$

Thus $\mathbb{H}$ is associative but not commutative. $\mathbb{R}$ and $\mathbb{C}$ are naturally embedded into $\mathbb{H}$ as follows:

$$\mathbb{R} = \mathbb{R} \cdot 1 \subset \mathbb{H}, \quad \mathbb{C} = \mathbb{R} \cdot 1 \oplus \mathbb{R} \cdot \mathrm{i} \subset \mathbb{H},$$

and we sometimes express an element of $\mathbb{H}$ as $q = z + w\mathrm{j}$, where $z, w \in \mathbb{C}$.

Conjugation is defined for quaternions:

$$\overline{a + b\mathrm{i} + c\mathrm{j} + d\mathrm{k}} = a - b\mathrm{i} - c\mathrm{j} - d\mathrm{k}\,(a,b,c,d \in \mathbb{R}).$$

Or equivalently,

$$\overline{z + w\mathrm{j}} = \bar{z} - w\mathrm{j}\,(z,w \in \mathbb{C}).$$

Then we have $\overline{pq} = \bar{q}\bar{p}$, for any $p, q \in \mathbb{H}$.

Let $\mathbb{H}^n$ be the space of n-dimensional quaternion column vectors. We consider it as a right $\mathbb{H}$-module. If $\boldsymbol{p} = (p_1, \cdots, p_n)^{\mathrm{T}}, \boldsymbol{q} = (q_1, \cdots, q_n)^{\mathrm{T}} \in \mathbb{H}^n$, two inner products of $\boldsymbol{p}, \boldsymbol{q}$ are defined:

$$\langle \boldsymbol{p}, \boldsymbol{q} \rangle_{\mathbb{H}} = \sum_l \bar{p}_l q_l, \ \langle \boldsymbol{p}, \boldsymbol{q} \rangle_{\mathbb{R}} = \mathrm{Re}\langle \boldsymbol{p}, \boldsymbol{q} \rangle_{\mathbb{H}}.$$

It is easily verified that $\langle\, , \rangle_{\mathbb{R}}$ is just the usual Euclidean inner product if $\mathbb{H}^n$ is identified as $\mathbb{R}^{4n}$, and that the following properties hold:

$$\langle \boldsymbol{p}x, \boldsymbol{q}y \rangle_{\mathbb{H}} = \bar{x}\langle \boldsymbol{p}, \boldsymbol{q} \rangle_{\mathbb{H}} y,$$

$$\langle \boldsymbol{p}, \boldsymbol{q} \rangle_{\mathbb{H}} = \overline{\langle \boldsymbol{q}, \boldsymbol{p} \rangle}_{\mathbb{H}},$$

where $\boldsymbol{p}, \boldsymbol{q} \in \mathbb{H}^n, x, y \in \mathbb{H}$.

Similarly, for $\boldsymbol{z} = (z_1, \cdots, z_n)^t, \boldsymbol{w} = (w_1, \cdots, w_n)^t \in \mathbb{C}^n$, we define their inner products:

$$\langle \boldsymbol{z}, \boldsymbol{w} \rangle_{\mathbb{C}} = \sum_l \bar{z}_l w_l, \ \langle \boldsymbol{z}, \boldsymbol{w} \rangle_{\mathbb{R}} = \mathrm{Re}\langle \boldsymbol{z}, \boldsymbol{w} \rangle_{\mathbb{C}}.$$

We will often omit the subscripts $\mathbb{C}$ and $\mathbb{H}$ for simplicity.

Next we consider the quaternion projective space $\mathbb{H}P^n$, the set of quaternionic lines in $\mathbb{H}^{n+1}$. Equivalently, $\mathbb{H}P^n = S^{4n+3}/Sp(1)$, where S^{4n+3} is the unit sphere in $\mathbb{H}^{n+1} \cong \mathbb{R}^{4n+4}$, and $Sp(1)$, the multiplicative group of unit quaternions, acts on S^{4n+3} by right multiplication. Since this is an isometric action, there is a unique Riemannian metric on $\mathbb{H}P^n$, called the Fubini-Study metric, such that the quotient map $\tau: S^{4n+3} \to \mathbb{H}P^n$ is a Riemannian submersion. For any $\boldsymbol{q} \in S^{4n+3}$, let $H_{\boldsymbol{q}}$ be the horizontal space of τ at $\boldsymbol{q}$, i. e. the normal space to the fibre $\tau^{-1}(\tau(\boldsymbol{q}))$. Then $H_{\boldsymbol{q}} = \{\boldsymbol{q}' \in \mathbb{H}^{n+1} \mid \langle \boldsymbol{q}, \boldsymbol{q}' \rangle_{\mathbb{H}} = 0\}$. Let $\tau_{\boldsymbol{q}} = \mathrm{d}\tau_{\boldsymbol{q}}|_{H_{\boldsymbol{q}}}$. By assumption, $\tau_{\boldsymbol{q}}: H_{\boldsymbol{q}} \to T_{\tau(\boldsymbol{q})}\mathbb{H}P^n$ is a linear isometry.

2 The submanifold $Q^{n+1} \subset \mathbb{H}P^{n+1}$; the complex Grassmannian $G(2, n+2)$

We quote some results from Ref. [4].

$SU(n+2)$ acts on $S^{4n+7} \subset \mathbb{H}^{n+2}$ isometrically via

$$SU(n+2) \times S^{4n+7} \to S^{4n+7},$$

$$(\boldsymbol{A}, \boldsymbol{z} + \boldsymbol{v}\mathrm{j}) \mapsto \boldsymbol{A}\boldsymbol{z} + (\boldsymbol{A}\boldsymbol{v})\mathrm{j},$$

where $\boldsymbol{z}, \boldsymbol{v} \in \mathbb{C}^{n+2}$, with $|\boldsymbol{z}|^2 + |\boldsymbol{v}|^2 = 1$. This action commutes with the $Sp(1)$-action on S^{4n+7} defined in the last section, hence descends to an isometric action on $\mathbb{H}P^{n+1}$.

By some straightforward calculations, we find that this $SU(n+2)$-action on $\mathbb{H}P^{n+1}$ has only two singular orbits, namely,

$$\mathbb{C}P^{n+1} = \{\tau(\boldsymbol{z} + 0 \cdot \mathrm{j}) \mid \boldsymbol{z} \in S^{2n+3}\}, \quad (1)$$

and

$$Q^{n+1} = \{\tau((1/\sqrt{2})(\boldsymbol{z} + \boldsymbol{v}\mathrm{j})) \mid \boldsymbol{z}, \boldsymbol{v} \in S^{2n+3}, \langle \boldsymbol{z}, \boldsymbol{v} \rangle = 0\}, \quad (2)$$

where S^{2n+3} is the unit sphere of $\mathbb{C}^{n+2}$.

We have the following proposition from Ref. [4]:

Proposition 2.1 The singular orbits of the $SU(n+2)$-action on $\mathbb{H}P^{n+1}$ are $\mathbb{C}P^{n+1}$ and Q^{n+1}. Q^{n+1} has codimension 3 in $\mathbb{H}P^{n+1}$, and is isometric to the homogeneous space $SU(n+2)/SU(2)\times SU(n)$ equipped with a suitable invariant metric. Furthermore, Q^{n+1} is a minimal submanifold of $\mathbb{H}P^{n+1}$.

Now consider an action of $U(1)$ on Q^{n+1}:

$$U(1)\times Q^{n+1}\to Q^{n+1},\ (e^{it},\tau(\boldsymbol{q}))\mapsto\tau(e^{it}\boldsymbol{q}),$$

where $t\in\mathbb{R}$. Again this is an isometric action. A vector field $\boldsymbol{\xi}$ on Q^{n+1} is defined:

$$\boldsymbol{\xi}_{\tau(q)}=\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0}e^{it}\cdot\tau(\boldsymbol{q})=\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0}\tau(e^{it}\boldsymbol{q})$$
$$=\mathrm{d}\tau_q(\mathrm{i}\boldsymbol{q})=\tau_q(\mathrm{i}\boldsymbol{q}). \tag{3}$$

Here $\boldsymbol{q}=\frac{1}{\sqrt{2}}(\boldsymbol{z}+\boldsymbol{v}\mathrm{j})\in\tau^{-1}(Q^{n+1}),\boldsymbol{z},\boldsymbol{v}\in S^{2n+3},\langle\boldsymbol{z},\boldsymbol{v}\rangle=0$. For the last equality, we note that $\langle\boldsymbol{q},\mathrm{i}\boldsymbol{q}\rangle_{\mathbb{H}}=0$ for $\boldsymbol{q}\in\tau^{-1}(Q^{n+1})$, i.e., $\mathrm{i}\boldsymbol{q}\in H_q$. Thus $\boldsymbol{\xi}$ is the field of tangent vectors to the orbits of the $U(1)$-action.

Let $B^{n+1}=Q^{n+1}/U(1)$. Since $U(1)$ acts on Q^{n+1} isometrically, there is a unique Riemannian metric on B^{n+1} such that the natural projection $\pi: Q^{n+1}\to B^{n+1}$ is a Riemannian submersion.

For $\tau(\boldsymbol{q})\in Q^{n+1}$, let $\mathcal{H}_{\tau(q)}$ be the orthogonal complement of $\boldsymbol{\xi}_{\tau(q)}$ in $T_{\tau(q)}Q^{n+1}$, i.e. the horizontal space of the Riemannian submersion $\pi: Q^{n+1}\to B^{n+1}$. Then the map $\pi_{\tau(q)}=\mathrm{d}\pi_{\tau(q)}|_{\mathcal{H}_{\tau(q)}}:\mathcal{H}_{\tau(q)}\to T_{\pi(\tau(q))}B^{n+1}$ is a linear isometry. By Ref. [4], the horizontal lift of $\mathcal{H}_{\tau(q)}$ through $\tau: S^{4n+7}\to\mathbb{H}P^{n+1}$ is

$$\tau_q^{-1}\mathcal{H}_{\tau(q)}=\{\boldsymbol{X}\in\mathbb{H}^{n+2}\mid\langle\boldsymbol{X},\boldsymbol{q}\rangle=\langle\boldsymbol{X},\mathrm{i}\boldsymbol{q}\rangle=0\}. \tag{4}$$

We define a $(1,1)$-tensor φ on Q^{n+1} as $\varphi X=-\nabla_X^{Q^{n+1}}\boldsymbol{\xi},X\in TQ^{n+1}$, where $\nabla^{Q^{n+1}}$ is the Riemannian connection on Q^{n+1}. Using the O'Neil formula for Riemannian submersions (see, for example, Proposition 4.5.1 of Ref. [6]), it can be shown that

$$\varphi X=\begin{cases}0, & X=\boldsymbol{\xi},\\ \tau_q(-\mathrm{i}\cdot\tau_q^{-1}(\boldsymbol{X})), & \boldsymbol{X}\in\mathcal{H}_{\tau(q)}.\end{cases} \tag{5}$$

Since by definition $T_{\tau(q)}Q^{n+1}=\mathbb{R}\cdot\boldsymbol{\xi}_{\tau(q)}\oplus\mathcal{H}_{\tau(q)}$, this completely determines φ. In particular, $\varphi(\mathcal{H})\subset\mathcal{H}$.

Finally, notice that φ commutes with the $U(1)$-action on Q^{n+1}. In other words, if L_t denotes the map $Q^{n+1}\to Q^{n+1},\tau(\boldsymbol{q})\mapsto\tau(e^{it}\boldsymbol{q})$, then $\mathrm{d}L_t\circ\varphi=\varphi\circ\mathrm{d}L_t$ for all $t\in\mathbb{R}$. Therefore, there exists a $(1,1)$-tensor J on B^{n+1} satisfying $J\pi_*=\pi_*\varphi$. As $\varphi^2\boldsymbol{X}=-\boldsymbol{X}$ for all $\boldsymbol{X}\in\mathcal{H}$, it follows that J is an almost Hermitian structure on B^{n+1}. Actually, as is proved in Ref. [4], J turns out to be Kähler, and B^{n+1} is holomorphically isometric to the complex Grassmannian

$$G(2,n+2)=U(n+2)/U(2)\times U(n),$$

where the metric on $G(2,n+2)$ is induced by the following bi-invariant metric on $U(n+2)$:

$$\langle\boldsymbol{X},\boldsymbol{Y}\rangle=-\frac{1}{4}\mathrm{tr}(\boldsymbol{XY}),(\boldsymbol{X},\boldsymbol{Y}\in\mathfrak{u}(n+2)).$$

Thus, for example, B^2 is isometric to $G(2,3)=\mathbb{C}P^2$, with the Fubini-Study metric of constant holomorphic sectional curvature 8.

Remark The isometry between $G(2,n+2)$ and B^{n+1} can be explicitly given as

$$G(2,n+2)\to B^{n+1},$$
$$\mathbb{C}\boldsymbol{z}\oplus\mathbb{C}\boldsymbol{v}\mapsto\pi(\tau(\frac{1}{\sqrt{2}}(\boldsymbol{z}+\boldsymbol{v}\mathrm{j}))),$$

where $\boldsymbol{z},\boldsymbol{v}\in\mathbb{C}^{n+2},|\boldsymbol{z}|=|\boldsymbol{v}|=1,\langle\boldsymbol{z},\boldsymbol{v}\rangle_{\mathbb{C}}=0$.

3 The main theorem

Definition 3.1 Suppose N is a Hermitian manifold, J is its complex structure, $f: M\to N$ is an immersion from a surface M to N. Then f is called totally real if $J\,\mathrm{Im}f_{*p}\perp\mathrm{Im}f_{*p}$ for all $p\in M$.

If we choose a local frame $\boldsymbol{X},\boldsymbol{Y}$ for M, then f is totally real if and only if $Jf_*\boldsymbol{X}\perp f_*\boldsymbol{Y}$ everywhere. This follows easily from the Hermitian condition $\langle J\boldsymbol{u},J\boldsymbol{v}\rangle=\langle\boldsymbol{u},\boldsymbol{v}\rangle,J^2=-1$, where $\langle,\rangle$ is the Riemannian metric on N.

Now we can state our main result.

Theorem 3.1 Suppose M is a surface, $\psi: M\to B^{n+1}$ an immersion, then the following are equivalent:

1) ψ is totally real;

2) ψ has local horizontal lifts to Q^{n+1}, that is,

for any $p \in M$, there is a neighborhood U of p, and an immersion $\eta: U \to Q^{n+1}$, such that $\pi \circ \eta = \psi$, and $\mathrm{Im}\, \eta_* \subset \mathcal{H}$.

Furthermore, η is minimal in Q^{n+1} if and only if ψ is minimal in B^{n+1}.

We prove the theorem step by step.

Step 1 Let U be an open subset of M, $\eta: U \to Q^{n+1}$ an immersion, we shall find a sufficient and necessary condition for η to be horizontal.

First, since $\tau: S^{4n+7} \to \mathbb{H}P^{n+1}$ is a submersion, η can be lifted to S^{4n+7}, that is, there is an immersion $\boldsymbol{q} = \frac{1}{\sqrt{2}}(\boldsymbol{Z} + \boldsymbol{V}\mathrm{j}): U \to \tau^{-1}(Q^{n+1}) \subset S^{4n+7}$ such that $\eta = \tau \circ q$, where $\boldsymbol{Z}, \boldsymbol{V}: U \to \mathbb{C}^{n+2}$, $|\boldsymbol{Z}| = |\boldsymbol{V}| = 1$, $\langle \boldsymbol{Z}, \boldsymbol{V} \rangle = 0$. Now

$$\begin{aligned} \mathrm{d}\eta &= \mathrm{d}\tau \mathrm{d}\boldsymbol{q} \\ &= \mathrm{d}\tau(\mathrm{d}\boldsymbol{q} - \boldsymbol{q}\langle \boldsymbol{q}, \mathrm{d}\boldsymbol{q} \rangle), \end{aligned} \tag{6}$$

so the horizontal lift of $\mathrm{d}\eta$ to S^{4n+7} is $\tau_q^{-1}\mathrm{d}\eta = \mathrm{d}\boldsymbol{q} - \boldsymbol{q}\langle \boldsymbol{q}, \mathrm{d}\boldsymbol{q} \rangle$, namely the orthogonal projection of $\mathrm{d}\boldsymbol{q}$ onto $H_{\boldsymbol{q}}$, the horizontal space of τ at $\boldsymbol{q}$.

Recall from the last section that

η is horizontal with respect to π

$\Leftrightarrow \mathrm{Im}(\mathrm{d}\eta) \subset \mathcal{H}$

$\Leftrightarrow \langle \tau_q^{-1}\mathrm{d}\eta, \boldsymbol{q} \rangle = \langle \tau_q^{-1}\mathrm{d}\eta, \mathrm{i}\boldsymbol{q} \rangle = 0$

$\Leftrightarrow \langle \mathrm{d}\boldsymbol{q} - \boldsymbol{q}\langle \boldsymbol{q}, \mathrm{d}\boldsymbol{q} \rangle, \mathrm{i}\boldsymbol{q} \rangle = 0$

$\Leftrightarrow \langle \mathrm{d}\boldsymbol{q}, \mathrm{i}\boldsymbol{q} \rangle = 0.$

For the last equivalence note that $\boldsymbol{q} \in \tau^{-1}(Q^{n+1})$ implies $\langle \boldsymbol{q}, \mathrm{i}\boldsymbol{q} \rangle = 0$.

Write $\boldsymbol{q} = \frac{1}{\sqrt{2}}(\boldsymbol{Z} + \boldsymbol{V}\mathrm{j})$, $\mathrm{d}\boldsymbol{q} = \frac{1}{\sqrt{2}}(\mathrm{d}\boldsymbol{Z} + \mathrm{d}\boldsymbol{V} \cdot \mathrm{j})$.

Differentiating $\langle \boldsymbol{V}, \boldsymbol{V} \rangle = 1$, $\langle \boldsymbol{Z}, \boldsymbol{V} \rangle = 0$ gives

$$\begin{cases} \langle \mathrm{d}\boldsymbol{V}, \boldsymbol{V} \rangle + \langle \boldsymbol{V}, \mathrm{d}\boldsymbol{V} \rangle = 0, \\ \langle \mathrm{d}\boldsymbol{Z}, \boldsymbol{V} \rangle + \langle \boldsymbol{Z}, \mathrm{d}\boldsymbol{V} \rangle = 0. \end{cases}$$

Then

$$\begin{aligned} &\langle \mathrm{d}\boldsymbol{q}, \mathrm{i}\boldsymbol{q} \rangle = 0 \\ \Leftrightarrow 0 &= \langle \mathrm{d}\boldsymbol{Z} + \mathrm{d}\boldsymbol{V} \cdot \mathrm{j}, \boldsymbol{Z}\mathrm{i} + \boldsymbol{V}\mathrm{k} \rangle \\ &= (\langle \mathrm{d}\boldsymbol{Z}, \boldsymbol{Z} \rangle - \langle \boldsymbol{V}, \mathrm{d}\boldsymbol{V} \rangle)\mathrm{i} + \\ &\quad (\langle \boldsymbol{Z}, \mathrm{d}\boldsymbol{V} \rangle + \langle \mathrm{d}\boldsymbol{Z}, \boldsymbol{V} \rangle)\mathrm{k} \\ &= (\langle \mathrm{d}\boldsymbol{Z}, \boldsymbol{Z} \rangle + \langle \mathrm{d}\boldsymbol{V}, \boldsymbol{V} \rangle)\mathrm{i}. \end{aligned}$$

In summary, we have proved

Lemma 3.1 Suppose $\eta = \tau(\frac{1}{\sqrt{2}}(\boldsymbol{Z} + \boldsymbol{V}\mathrm{j})): U \to Q^{n+1}$ is an immersion. Then η is horizontal with respect to $\pi: Q^{n+1} \to B^{n+1}$ if and only if

$$\langle \mathrm{d}\boldsymbol{Z}, \boldsymbol{Z} \rangle + \langle \mathrm{d}\boldsymbol{V}, \boldsymbol{V} \rangle = 0. \tag{7}$$

Step 2 Let $\psi: M \to B^{n+1}$ be an immersion of a surface M into B^{n+1}. We look for the condition under which ψ has a local horizontal lift to Q^{n+1}.

Let $\eta = \tau \circ \boldsymbol{q} = \tau(\frac{1}{\sqrt{2}}(\boldsymbol{Z} + \boldsymbol{V}\mathrm{j})): U \to Q^{n+1}$ be any local lift of ψ, and η_0 a horizontal lift of ψ. Recall that B^{n+1} is defined as the quotient of Q^{n+1} under the $U(1)$-action $\mathrm{e}^{\mathrm{i}t} \cdot \tau(\boldsymbol{q}) = \tau(\mathrm{e}^{\mathrm{i}t}\boldsymbol{q})$, then for any $p \in U$, $\eta(p)$ and $\eta_0(p)$ lie in the same orbit. It follows that there is a map $\lambda: U \to U(1)$ such that $\eta_0(p) = \lambda(p) \cdot \eta(p)$ for all $p \in U$. In short,

$$\eta_0 = \lambda \cdot \eta = \tau(\frac{1}{\sqrt{2}}(\lambda\boldsymbol{Z} + \lambda\boldsymbol{V}\mathrm{j})). \tag{8}$$

Since η_0 is horizontal, we apply Lemma 1 to obtain

$$\begin{aligned} 0 &= \langle \mathrm{d}(\lambda\boldsymbol{Z}), \lambda\boldsymbol{Z} \rangle + \langle \mathrm{d}(\lambda\boldsymbol{V}), \lambda\boldsymbol{V} \rangle \\ &= \langle \mathrm{d}\lambda \cdot \boldsymbol{Z} + \lambda\mathrm{d}\boldsymbol{Z}, \lambda\boldsymbol{Z} \rangle + \langle \mathrm{d}\lambda \cdot \boldsymbol{V} + \lambda\mathrm{d}\boldsymbol{V}, \lambda\boldsymbol{V} \rangle \\ &= \lambda\mathrm{d}\bar{\lambda}(\langle \boldsymbol{Z}, \boldsymbol{Z} \rangle + \langle \boldsymbol{V}, \boldsymbol{V} \rangle) + \lambda\bar{\lambda}(\langle \mathrm{d}\boldsymbol{Z}, \boldsymbol{Z} \rangle + \langle \mathrm{d}\boldsymbol{V}, \boldsymbol{V} \rangle) \\ &= -2\bar{\lambda}\mathrm{d}\lambda + \langle \mathrm{d}\boldsymbol{Z}, \boldsymbol{Z} \rangle + \langle \mathrm{d}\boldsymbol{V}, \boldsymbol{V} \rangle. \end{aligned}$$

Here we have used $\lambda\bar{\lambda} = 1$ and $\bar{\lambda}\mathrm{d}\lambda + \lambda\mathrm{d}\bar{\lambda} = 0$.

Since $\bar{\lambda}\mathrm{d}\lambda = \lambda^{-1}\mathrm{d}\lambda = \mathrm{d}(\log\lambda)$ we get

$$2\mathrm{d}(\log\lambda) = \langle \mathrm{d}\boldsymbol{Z}, \boldsymbol{Z} \rangle + \langle \mathrm{d}\boldsymbol{V}, \boldsymbol{V} \rangle. \tag{9}$$

If we take a local coordinate (x, y) on M, this amounts to

$$\begin{cases} 2\dfrac{\partial \log\lambda}{\partial x} = \langle \boldsymbol{Z}_x, \boldsymbol{Z} \rangle + \langle \boldsymbol{V}_x, \boldsymbol{V} \rangle, \\ 2\dfrac{\partial \log\lambda}{\partial y} = \langle \boldsymbol{Z}_y, \boldsymbol{Z} \rangle + \langle \boldsymbol{V}_y, \boldsymbol{V} \rangle, \end{cases} \tag{10}$$

where $\boldsymbol{Z}_x = \frac{\partial \boldsymbol{Z}}{\partial x}$, $\boldsymbol{Z}_y = \frac{\partial \boldsymbol{Z}}{\partial y}$, etc. This is a system of first-order PDEs in λ. By the Frobenius theorem for PDEs, an initial value problem of such a system is solvable if and only if the integrability condition

$$\frac{\partial}{\partial y}\left(\frac{\partial \log\lambda}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial \log\lambda}{\partial y}\right),$$

that is,

$$\frac{\partial}{\partial y}(\langle \boldsymbol{Z}_x, \boldsymbol{Z} \rangle + \langle \boldsymbol{V}_x, \boldsymbol{V} \rangle) = \frac{\partial}{\partial x}(\langle \boldsymbol{Z}_y, \boldsymbol{Z} \rangle + \langle \boldsymbol{V}_y, \boldsymbol{V} \rangle)$$

holds. This equation simplifies to

$$\langle \boldsymbol{Z}_x, \boldsymbol{Z}_y \rangle + \langle \boldsymbol{V}_x, \boldsymbol{V}_y \rangle = \langle \boldsymbol{Z}_y, \boldsymbol{Z}_x \rangle + \langle \boldsymbol{V}_y, \boldsymbol{V}_x \rangle. \tag{11}$$

Thus we obtain

Lemma 3.2 Suppose $\psi = \pi \circ \tau(\frac{1}{\sqrt{2}}(\boldsymbol{Z} + \boldsymbol{V}\mathrm{j})) : M \to B^{n+1}$ is an immersion. Then ψ has local horizontal lifts to Q^{n+1} if and only if (11) holds.

Step 3 Let $\psi = \pi \circ \tau \circ q : M \to B^{n+1}$, where $q = \frac{1}{\sqrt{2}}(\boldsymbol{Z} + \boldsymbol{V}\mathrm{j}) : M \to \tau^{-1}(Q^{n+1}) \subset S^{4n+7}$. We shall find out the equation for ψ to be totally real.

We have

$$\begin{aligned} \mathrm{d}\psi &= \mathrm{d}\pi\mathrm{d}\tau\mathrm{d}\boldsymbol{q} \\ &= \mathrm{d}\pi\mathrm{d}\tau(\mathrm{d}\boldsymbol{q} - \boldsymbol{q}\langle \boldsymbol{q}, \mathrm{d}\boldsymbol{q}\rangle - \mathrm{i}\boldsymbol{q}\langle \mathrm{i}\boldsymbol{q}, \mathrm{d}\boldsymbol{q}\rangle) \qquad (12) \\ &= \mathrm{d}\pi\mathrm{d}\tau(\mathrm{d}\boldsymbol{q}^{\mathcal{H}}), \end{aligned}$$

where $\mathrm{d}\boldsymbol{q}^{\mathcal{H}} = \mathrm{d}\boldsymbol{q} - \boldsymbol{q}\langle \boldsymbol{q}, \mathrm{d}\boldsymbol{q}\rangle - \mathrm{i}\boldsymbol{q}\langle \mathrm{i}\boldsymbol{q}, \mathrm{d}\boldsymbol{q}\rangle$ is the orthogonal projection of $\mathrm{d}q$ onto $\tau_{\boldsymbol{q}}^{-1}\mathcal{H}_{\tau(\boldsymbol{q})}$. In other words, $\mathrm{d}\boldsymbol{q}^{\mathcal{H}} = \tau_{\boldsymbol{q}}^{-1}\pi_{\tau(\boldsymbol{q})}^{-1}(\mathrm{d}\psi)$.

Choose a local coordinate (x, y) on M. Then, using the definitions of the tensors φ, J (see (5)), and the fact that τ, π are Riemannian submersions, we obtain

$$\begin{aligned} &\psi \text{ is totally real} \\ \Leftrightarrow 0 &= \langle \psi_x, J\psi_y\rangle_{B^{n+1}} \\ &= \langle \pi_{\tau(\boldsymbol{q})}^{-1}\psi_x, \pi_{\tau(\boldsymbol{q})}^{-1}J\psi_y\rangle_{Q^{n+1}} \\ &= \langle \pi_{\tau(\boldsymbol{q})}^{-1}\psi_x, \varphi\pi_{\tau(\boldsymbol{q})}^{-1}\psi_y\rangle_{Q^{n+1}} \\ &= \langle \tau_{\boldsymbol{q}}^{-1}\pi_{\tau(\boldsymbol{q})}^{-1}\psi_x, \tau_{\boldsymbol{q}}^{-1}\varphi\pi_{\tau(\boldsymbol{q})}^{-1}\psi_y\rangle_{\mathbb{R}} \\ &= \langle \tau_{\boldsymbol{q}}^{-1}\pi_{\tau(\boldsymbol{q})}^{-1}\psi_x, -\mathrm{i}\cdot\tau_{\boldsymbol{q}}^{-1}\pi_{\tau(\boldsymbol{q})}^{-1}\psi_y\rangle_{\mathbb{R}} \\ &= \langle \boldsymbol{q}_x^{\mathcal{H}}, -\mathrm{i}\boldsymbol{q}_y^{\mathcal{H}}\rangle_{\mathbb{R}}. \qquad (13) \end{aligned}$$

Since $\langle \boldsymbol{q}_x^{\mathcal{H}}, \mathrm{i}\boldsymbol{q}_y^{\mathcal{H}}\rangle_{\mathbb{R}} = \mathrm{Re}\langle \boldsymbol{q}_x^{\mathcal{H}}, \mathrm{i}\boldsymbol{q}_y^{\mathcal{H}}\rangle$, let us calculate $\langle \boldsymbol{q}_x^{\mathcal{H}}, \mathrm{i}\boldsymbol{q}_y^{\mathcal{H}}\rangle$ first. Now

$$\begin{aligned} &\langle \boldsymbol{q}_x^{\mathcal{H}}, \mathrm{i}\boldsymbol{q}_y^{\mathcal{H}}\rangle \\ &= \langle \boldsymbol{q}_x - \boldsymbol{q}\langle \boldsymbol{q}, \boldsymbol{q}_x\rangle - \mathrm{i}\boldsymbol{q}\langle \mathrm{i}\boldsymbol{q}, \boldsymbol{q}_x\rangle, \mathrm{i}\boldsymbol{q}_y - \mathrm{i}\boldsymbol{q}\langle \boldsymbol{q}, \boldsymbol{q}_y\rangle + \boldsymbol{q}\langle \mathrm{i}\boldsymbol{q}, \boldsymbol{q}_y\rangle\rangle \\ &= \langle \boldsymbol{q}_x, \mathrm{i}\boldsymbol{q}_y\rangle - \cancel{\langle \boldsymbol{q}_x, \mathrm{i}\boldsymbol{q}\rangle\langle \boldsymbol{q}, \boldsymbol{q}_y\rangle} + \cancel{\langle \boldsymbol{q}_x, \boldsymbol{q}\rangle\langle \mathrm{i}\boldsymbol{q}, \boldsymbol{q}_y\rangle} - \\ &\quad \langle \boldsymbol{q}_x, \boldsymbol{q}\rangle\langle \boldsymbol{q}, \mathrm{i}\boldsymbol{q}_y\rangle - \cancel{\langle \boldsymbol{q}_x, \mathrm{i}\boldsymbol{q}\rangle\langle \mathrm{i}\boldsymbol{q}, \boldsymbol{q}_y\rangle} - \\ &\quad \langle \boldsymbol{q}_x, \mathrm{i}\boldsymbol{q}\rangle\langle \mathrm{i}\boldsymbol{q}, \mathrm{i}\boldsymbol{q}_y\rangle + \cancel{\langle \boldsymbol{q}_x, \mathrm{i}\boldsymbol{q}\rangle\langle \boldsymbol{q}, \boldsymbol{q}_y\rangle} \\ &= \langle \boldsymbol{q}_x, \mathrm{i}\boldsymbol{q}_y\rangle - \langle \boldsymbol{q}_x, \boldsymbol{q}\rangle\langle \boldsymbol{q}, \mathrm{i}\boldsymbol{q}_y\rangle - \langle \boldsymbol{q}_x, \mathrm{i}\boldsymbol{q}\rangle\langle \mathrm{i}\boldsymbol{q}, \mathrm{i}\boldsymbol{q}_y\rangle \end{aligned}$$

For the second step note that $\boldsymbol{q} \in \tau^{-1}(Q^{n+1})$ implies $\langle \boldsymbol{q}, \mathrm{i}\boldsymbol{q}\rangle = 0$. Differentiating $\langle \boldsymbol{q}, \boldsymbol{q}\rangle = 1$ yields

$$\begin{aligned} 0 &= \langle \boldsymbol{q}_x, \boldsymbol{q}\rangle + \langle \boldsymbol{q}, \boldsymbol{q}_x\rangle \\ &= \langle \boldsymbol{q}_x, \boldsymbol{q}\rangle + \overline{\langle \boldsymbol{q}_x, \boldsymbol{q}\rangle}, \end{aligned}$$

i. e., $\langle \boldsymbol{q}_x, \boldsymbol{q}\rangle \in \mathrm{Im}\mathbb{H}$. Similarly, differentiating $\langle \boldsymbol{q}, \mathrm{i}\boldsymbol{q}\rangle = 0$ yields

$$\begin{aligned} 0 &= \langle \boldsymbol{q}_y, \mathrm{i}\boldsymbol{q}\rangle + \langle \boldsymbol{q}, \mathrm{i}\boldsymbol{q}_y\rangle \\ &= -\langle \mathrm{i}\boldsymbol{q}_y, \boldsymbol{q}\rangle + \langle \boldsymbol{q}, \mathrm{i}\boldsymbol{q}_y\rangle \\ &= -\overline{\langle \boldsymbol{q}, \mathrm{i}\boldsymbol{q}_y\rangle} + \langle \boldsymbol{q}, \mathrm{i}\boldsymbol{q}_y\rangle, \end{aligned}$$

i. e., $\langle \boldsymbol{q}, \mathrm{i}\boldsymbol{q}_y\rangle \in \mathbb{R}$. Therefore $\langle \boldsymbol{q}_x, \boldsymbol{q}\rangle\langle \boldsymbol{q}, \mathrm{i}\boldsymbol{q}_y\rangle \in \mathrm{Im}\mathbb{H}$. Similarly $\langle \boldsymbol{q}_x, \mathrm{i}\boldsymbol{q}\rangle\langle \mathrm{i}\boldsymbol{q}, \mathrm{i}\boldsymbol{q}_y\rangle \in \mathrm{Im}\mathbb{H}$. Thus we get

$$\begin{aligned} &2\langle \boldsymbol{q}_x^{\mathcal{H}}, \mathrm{i}\boldsymbol{q}_y^{\mathcal{H}}\rangle_{\mathbb{R}} \\ &= 2\mathrm{Re}\langle \boldsymbol{q}_x^{\mathcal{H}}, \mathrm{i}\boldsymbol{q}_y^{\mathcal{H}}\rangle \\ &= 2\mathrm{Re}\langle \boldsymbol{q}_x, \mathrm{i}\boldsymbol{q}_y\rangle \\ &= \mathrm{Re}\langle \boldsymbol{Z}_x + \boldsymbol{V}_x\mathrm{j}, \boldsymbol{Z}_y\mathrm{i} + \boldsymbol{V}_y\mathrm{k}\rangle \\ &= \mathrm{Re}(\langle \boldsymbol{Z}_x, \boldsymbol{Z}_y\rangle\mathrm{i} - \langle \boldsymbol{V}_y, \boldsymbol{V}_x\rangle\mathrm{i}) \\ &= \mathrm{Im}(\langle \boldsymbol{V}_y, \boldsymbol{V}_x\rangle - \langle \boldsymbol{Z}_x, \boldsymbol{Z}_y\rangle). \qquad (14) \end{aligned}$$

Finally, from (13) and (14) we obtain

Lemma 3.3 $\psi = \pi \circ \tau(\frac{1}{\sqrt{2}}(\boldsymbol{Z} + \boldsymbol{V}\mathrm{j})) : M \to B^{n+1}$ is totally real if and only if

$$\mathrm{Im}(\langle \boldsymbol{V}_y, \boldsymbol{V}_x\rangle - \langle \boldsymbol{Z}_x, \boldsymbol{Z}_y\rangle) = 0,$$

or equivalently,

$$\langle \boldsymbol{V}_y, \boldsymbol{V}_x\rangle - \langle \boldsymbol{Z}_x, \boldsymbol{Z}_y\rangle = \langle \boldsymbol{V}_x, \boldsymbol{V}_y\rangle - \langle \boldsymbol{Z}_y, \boldsymbol{Z}_x\rangle. \qquad (15)$$

Comparing with Lemma 3.2, we find that ψ have a local horizontal lift to Q^{n+1} if and only if it is totally real.

Step 4 We need a simple lemma.

Lemma 3.4 Suppose $\pi: \bar{N} \to N$ is a Riemannian submersion, $\bar{M} \subset \bar{N}$ is a horizontal submanifold, and $M = \pi(\bar{M}) \subset N$. Then

$$\boldsymbol{H}_M(\pi(p)) = \pi_*(\boldsymbol{H}_{\bar{M}}(p))$$

for any $p \in \bar{M}$. Furthermore, $\boldsymbol{H}_{\bar{M}}$ is horizontal. Here $\boldsymbol{H}_M, \boldsymbol{H}_{\bar{M}}$ are the mean curvature vectors of M, $\bar{M}$, respectively.

Proof Let $\bar{\boldsymbol{e}}_1, \cdots, \bar{\boldsymbol{e}}_m$ be an orthonormal frame on $\bar{M}$, then, since $\pi|_{\bar{M}}: \bar{M} \to M$ is an isometry, $\boldsymbol{e}_1 = \pi_*(\bar{\boldsymbol{e}}_1), \ldots, \boldsymbol{e}_m = \pi_*(\bar{\boldsymbol{e}}_m)$ is an orthonormal frame on M. By O'Neil's formula, $\nabla^{\bar{N}}_{\bar{\boldsymbol{e}}_i}\bar{\boldsymbol{e}}_i$ is the horizontal lift of $\nabla^N_{\boldsymbol{e}_i}\boldsymbol{e}_i$, hence horizontal, and $\boldsymbol{B}_{\bar{M}}(\bar{\boldsymbol{e}}_i, \bar{\boldsymbol{e}}_i) = \nabla^{\bar{N}}_{\bar{\boldsymbol{e}}_i}\bar{\boldsymbol{e}}_i - \nabla^{\bar{M}}_{\bar{\boldsymbol{e}}_i}\bar{\boldsymbol{e}}_i$ is also horizontal. Thus $\boldsymbol{H}_{\bar{M}} = \sum_i \boldsymbol{B}_{\bar{M}}(\bar{\boldsymbol{e}}_i, \bar{\boldsymbol{e}}_i)$ is horizontal. On the other hand,

$$\begin{aligned}\nabla^N_{e_i} e_i &= \pi_*(\bar{\nabla}^N_{\bar{e}_i} \bar{e}_i) \\ &= \pi_*(\nabla^{\bar{M}}_{\bar{e}_i} \bar{e}_i + B_{\bar{M}}(\bar{e}_i, \bar{e}_i)) \\ &= \nabla^M_{e_i} e_i + \pi_*(B_{\bar{M}}(\bar{e}_i, \bar{e}_i)).\end{aligned} \tag{16}$$

Comparing with the Gauss equation in N, we find that

$$B_M(e_i, e_i) = \pi_*(B_{\bar{M}}(\bar{e}_i, \bar{e}_i)). \tag{17}$$

The conclusion follows immediately. □

From the above lemma, we see that $H_M = 0 \Leftrightarrow H_{\bar{M}} = 0$. That is, M minimal $\Leftrightarrow \bar{M}$ minimal. This applies to our situation and the main theorem is fully proved.

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